Constrained Optimal Control of Bilinear Systems using Neural Network Based HJB Solution

Dipak M. Adhyaru, I. N. Kar, Senior Member, IEEE and M. Gopal

Abstract-**In this paper, a Hamilton-Jacobi-Bellman (HJB) equation based optimal control algorithm is proposed for a bilinear system. Utilizing the Lyapunov direct method, the controller is shown to be optimal with respect to a cost functional, which includes penalty on the control effort and the system states. In the proposed algorithm, Neural Network (NN) is used to find approximate solution of HJB equation using least squares method. Proposed algorithm has been applied on bilinear systems. Necessary theoretical and simulation results are presented to validate proposed algorithm.**

I. Introduction

Linear models frequently utilized to approximate the dynamical behavior of nature's nonlinear processes. Though linear approximations are convenient, they are inadequate for many processes. There is a particular form of nonlinear systems, which is quite common in nature and some interesting properties have been obtained for it. These systems are termed as a bilinear system and they are linear in state and linear in control, but not jointly linear in both. In the recent years, bilinear systems have been fairly exhaustively studied. This interest is essentially due to the fact that numerous real-world dynamical plants enjoy a bilinear structure. Various applications of bilinear modeling found in the engineering area like nuclear, thermal and chemical processes. Important satisfactory results on the structural properties of bilinear systems are available in the literature. However feedback control and stability of these systems are not explored much. From the practical point of view there is a need for the application oriented controller design technique for bilinear systems. For a bilinear system with a standard quadratic cost functional, it is not possible to express the optimal control in the feedback form except for simple cases. The obtained optimal controls have problems with global stabilization of the closed-loop system. Gutmann [3] and Longchamp [20] derived stabilizing feedback controls for bilinear systems.

Dipak M. Adhyaru is with Department of Electrical Engineering, Indian Institute of Technology,Delhi,Hauz Khas, New Delhi-110016, INDIA email: hetdip@rediffmail.com.(Tel. No. +91-11-26591093, Fax No.+91-11-26581606)

I.N. Kar is faculty of the Department of Electrical Engineering, Indian Institute of Technology,Delhi,Hauz Khas, New Delhi-110016, INDIA email: ink@ee.iitd.ac.in

M. Gopal is faculty of the Department of Electrical Engineering, Indian Institute of Technology,Delhi,Hauz Khas, New Delhi-110016, INDIA email: mgopal@ee.iitd.ac.in

Hofer and Tibken [17] proposed an approximation procedure of optimal control which can be solved by considering the quadratic bilinear regulator problem as a sequence of linear regulator solutions. Leitmann at. Al. [7] proposed an optimal control of bilinear systems using well known Hamilton-Jacobi-Bellman (HJB) equation, which is difficult to solve. Zijad and Zoran proposed the successive approximation procedure for optimal control of bilinear systems using riccati equation. Recently similar concept extended with successive galerkin approximation method by kim and Lim [16]. In all the above mentioned algorithms, constraint on the control input was not taken into account. However, in practical systems one should consider it due to limitation of the actuators. On the other hand, finding the solution of the HJB equation for constrained optimal control is a challenging task. HJB based optimal controller design for linear and nonlinear system has been explored by many researchers [9], [10], [13]. Neural Network (NN) based optimal control [2] uses a technique to approximate value-function, which gives approximate solution of HJB equation. However, to design an NN based constrained optimal control using HJB solution, is not much explored for bilinear system. An alternative approach is to obtain approximate solution of HJB equation using NN for feedback controller design for bilinear systems with constraints on the inputs.

In this paper the constrained optimal control problem proposed for bilinear systems by properly choosing cost functional. The cost functional is modified to account for constraints on the input. Our main contribution is to realize this approach with necessary theoretical justifications and implementation of it through NN. We have used least squares method to find tuning law of neural network, which is used to approximate the solution of HJB equation. In addition to this we also explored proposed algorithm on two bilinear systems to ensure asymptotic stability of the closed loop system. Furthermore it is shown that the Lyapunov function guaranteeing stability is a solution to the HJB equation for the nominal system. Convergence proof in present work is supplemented by necessary theoretical and simulation results.

The paper is organized as follows: In section 2, optimal control framework has been described for bilinear systems using necessary theoretical results. In section 3 NN based HJB solution is used to find optimal control. Stability issues with this formulation are discussed. Theoretical results are presented in the form of a lemma and a theorem. Solution of NN based HJB equation found

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by least squares method in section 4. Numerical examples are given in section 5 for the validity of the approach explained in section 3. Proposed work is concluded in section 6.

II. Optimal Control Of Bilinear Systems

Consider a bilinear system

$$\dot{x} = A(x) + B(x)u + \{xN\}u \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, constant matrix $N \in \mathbb{R}^{n\times m}$ and $u \in \mathbb{R}^m$ is the control input bounded by $|u| \le \psi \in \mathbb{R}$. $A(x)$ and $B(x)$ are known with $A(0) = 0$. In this paper we seek a constrained optimal control from the HJB equation.

Problem Statement:

Find a feedback control $u = K(x)$ that minimizes the cost functional

$$\int_0^\infty (x^T Qx + M(u))dt \ .$$

where

$$\begin{aligned} M(u) &= 2\int_0^u \psi \tanh^{-1}(v/A)Rdv \\ &= 2\psi uR\tanh^{-1}(u/\psi) + \psi^2 R\ln(1-u^2/\psi^2) > 0 \end{aligned} \quad (2)$$

is nonquadratic term expressing cost related to constrained input, Q and R are positive definite matrices. In this paper, we addressed the following problems:

1. To find an optimal control using HJB equation for bilinear system.
2. Solve the optimal control problem using NN. More specifically, HJB equation is solved using Neural Network.

To solve the optimal control problem, let

$$V(x_0) = \min_u \int_0^\infty (x^T Qx + M(u))dt$$

be the minimum cost of bringing the system (1) from initial condition x_0 to equilibrium point 0. Assuming $V(x)$ is only function of x, HJB equation gives

$$\min_u (x^T Qx + M(u) + V_x^T (A(x) + B(x)u + \{xN\}u)) = 0 \quad (3)$$

where $V_x = \dfrac{\partial V(x)}{\partial x}$.

If $u = K(x)$ is the solution to the optimal control problem then according to the Bellman's optimality principle [18], it can be found by solving following HJB equation:

$$\begin{aligned} \mathrm{HJB}(V(x)) = x^T Qx + M(u) + V_x^T (A(x) \\ + B(x)K(x) + \{xN\}u)) = 0 \end{aligned} \quad (4)$$

The optimal control can be found by solving $\dfrac{\partial \mathrm{HJB}(V(x))}{\partial u} = 0$

$$\Rightarrow \frac{\partial M(u)}{\partial u} + V_x \left(B(x) + Nx\right)^T = 0$$

Using definition of $M(u)$ in (2) one can write above equation as

$$2\psi R \tanh^{-1}(u/\psi) + V_x \left(B(x) + Nx\right)^T = 0 \quad (5)$$

$$\Rightarrow 2\psi R \tanh^{-1}(-u/\psi) = V_x \left(B(x) + Nx\right)^T$$

$$\Rightarrow u = K(x) = -\psi \tanh\left(\frac{R^{-1}}{2\psi} V_x \left(B(x) + Nx\right)^T\right) \quad (6)$$

Here $V(x)$ is the optimum solution of HJB equation (3). With this basic introduction, following results stated to show the existence of an optimal control.

Theorem 1: If $u = K(x)$ is satisfies the HJB equation (3) then it is an optimal control of the bilinear system (1).

Proof: Here $u = K(x)$ is an optimal control defined by equation (6) and $V(x)$ is the optimum solution of HJB equation (3). We have to show that the equilibrium point $x = 0$ of system (1) is globally asymptotically stable for the control $u = K(x)$. To do this, we show that $V(x)$ is a Lyapunov function. Clearly,

$$V(x) > 0, \ \ x \ne 0$$
$$V(x) = 0, \ \ x = 0$$

Also, $\dot{V}(x) = \partial V/\partial t < 0$ for $x \ne 0$, because

$$\begin{aligned} \dot{V}(x) &= (\partial V/\partial x)^T (dx/dt) \\ &= V_x^T(x)\left(A(x) + B(x)K(x) + \{xN\}u\right) \\ &= V_x^T(x)(A(x) + B(x)K(x)) + V_x^T(x)B(x)f(x) \end{aligned}$$

Using equation (6) and (7),

$$\dot{V}(x) = -x^T Qx - M(u) \le -x^T x \le 0$$

Here $Q = I$ (Identity matrix) assumed for the simplicity.

Thus conditions for Lyapunov local stability theory are satisfied. Consequently, there exists a neighborhood $Z = \{x : \|x\| < p\}$ for some $p > 0$ such that if $x(t)$ enters Z , then $\lim_{t\to\infty} x(t) = 0$ .But $x(t)$ cannot remain forever outside Z . Otherwise,

$$\|x\| \ge p \ \text{ for all } t \ge 0 .$$

Let $x^T x = \alpha \ge 0$. It is some scalar quantity.

Therefore

$$\begin{aligned} V(x(t)) - V(x(0)) &= \int_0^t \dot{V}(x(\tau))d\tau \\ &\le -\int_0^t \alpha d\tau = -\alpha \int_0^t d\tau = -\alpha t \end{aligned}$$

Let $t \to \infty$, we have, $V(x(t)) \le V(x(0)) - \alpha t \to -\infty$

Which contradicts the fact that $V(x) > 0$ for $\forall\ x \ne 0$. Therefore $\lim_{t\to\infty} x(t) = 0$ no matter where the trajectory begins. □

Hence by knowing exact solution of HJB equation we can find optimal control of bilinear system. To find the solution of HJB equation is very difficult problem. In the next section NN is used to approximate value-function V which is the solution of HJB equation.

III. NN Based Optimal Control

In this section we used NN to find approximate solution of HJB equation, which is utilized to find optimal control of bilinear system. It is well known that a NN can be used to approximate the function on a prescribed compact set [19]. It can be used to approximate a nonlinear mapping. Let $\mathbb{R}$ denote the real numbers. Given $x_k \in \mathbb{R}$, define $x=[x_0,x_1,...x_n]^T$, $y=[y_0,y_1,...y_m]^T$ and weight matrices $W=[w_1\ w_2.....w_L]^T$. Then the ideal NN output can be expressed as $y = W^T\sigma(x)$ with the vector of NN activation functions $\sigma(x) = [\sigma_1(x), \sigma_2(x), ..., \sigma_L(x)]$. It is assumed to be orthonormal and satisfy the NN approximation property [15]. With this background we proposed the optimal control framework based on NN.

Let the structure of NN based approximate value function can be defined as

$$\hat{V}(x) = \sum_{j=1}^{L} w_j \sigma_j(x) = W^T \sigma(x) \tag{7}$$

where $W=[w_1\ w_2.....w_L]^T$ is the set of NN weights. $\sigma(x)$ is selected such that $\hat{V}(0)=0$ and $\hat{V}(x)>0$ for $\forall\ x \neq 0$. So,

$$\hat{V}_x(x) = \frac{\partial \hat{V}}{\partial x} = W^T \nabla \sigma(x) \tag{8}$$

HJB equation with this assumption is as follows:

$$\begin{aligned}\mathrm{HJB}(\hat{V}(x)) = x^T Q x + M(\hat{u}) + \hat{V}_x^T (A(x) \\ + B(x)\hat{u} + \{xN\}\hat{u}) = e\end{aligned} \tag{9}$$

Here NN is used to approximate value function $V(x)$. Approximation error is represented by e. If e is negligible, then (9) becomes similar to (4).

With reference to the problem defined in section 2, we have introduced supporting theoretical results in the form of a lemma and a theorem. Lemma 1 is introduced to show the existence of NN based HJB solution proved for optimal control using modified performance functional. Theorem 2 shows the existence of the optimal control using NN based HJB solution for bilinear systems.

Lemma 1: Let $\hat{V}(x) = \sum_{j=1}^{L} w_j \sigma_j(x)$ satisfy $\left\langle \mathrm{HJB}(\hat{V}(x)), \sigma(x) \right\rangle_\Omega = 0$ and $\left\langle \hat{V}(x), \sigma(x) \right\rangle_\Omega = 0$ on a compact set $\Omega \subset \mathbb{R}^n$, and let $V(x) = \sum_{j=1}^{\infty} c_j \sigma_j(x)$ and $C=[c_1\ c_2......c_L]^T$ satisfy HJB ($V(x)$) =0.

If Ω is compact, $x^T Q x$ is continuous on Ω and are in the space span $\{\sigma_j\}_1^\infty$, and if the coefficients $|w_j|$ are uniformly bounded for all L, then $\left|\mathrm{HJB}(\hat{V}(x))\right| \to 0$ uniformly on Ω as L increases.

Proof: Khalaf et. al. [14] proposed theorem for the existence of NN based HJB solution for the optimal control problem of nonlinear systems. The existence on NN based HJB solution for optimal control of bilinear systems can be proved in the same line of it. The hypothesis implies that $\mathrm{HJB}(\hat{V}(x))$ are in $L_2(\Omega)$.

$$\begin{aligned}\left\langle \mathrm{HJB}(\hat{V}(x)), \sigma_j(x) \right\rangle_\Omega &= \sum_{k=1}^{L} w_k \left\langle \nabla\sigma_k(x) A(x), \sigma_j(x) \right\rangle_\Omega \\ &+ \sum_{k=1}^{L} w_k \left\langle M(\hat{u}), \sigma_j(x) \right\rangle_\Omega + \left\langle \left(x^T Q x\right), \sigma_j(x) \right\rangle_\Omega \\ &- \sum_{k=1}^{L} \left\langle \begin{matrix} w_k \nabla\sigma_k(x) \left(B(x) + \{Nx\}\right)\psi \cdot \\ \tanh\left(\frac{R^{-1}}{2\psi} \left(B(x) + \{Nx\}\right)^T \nabla\sigma_L^T(x) W_L \right) \end{matrix}, \sigma_j(x) \right\rangle_\Omega\end{aligned} \tag{10}$$

Also,

$$\left|\mathrm{HJB}(\hat{V}(x))\right| = \left| \sum_{j=1}^{\infty} \left\langle \mathrm{HJB}(\hat{V}(x)), \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right|,$$

using (10) one can write,

$$\begin{aligned}\left|\mathrm{HJB}(\hat{V}(x))\right| = \Bigg| &\sum_{j=L+1}^{\infty} \left(\sum_{k=1}^{L} w_k \left\langle \nabla\sigma_k(x) A(x), \sigma_j(x) \right\rangle_\Omega \right) \cdot \sigma_j(x) \\ &+ \sum_{j=L+1}^{\infty} \left\langle \left(x^T Q x\right), \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) + \sum_{j=L+1}^{\infty} \left(\sum_{k=1}^{L} w_k \left\langle M(\hat{u}), \sigma_j(x) \right\rangle_\Omega \right) \cdot \sigma_j(x) \\ &- \sum_{j=L+1}^{\infty} \left(\sum_{k=1}^{L} \left\langle \begin{matrix} w_k \nabla\sigma_k(x) \left(B(x) + \{Nx\}\right)\psi \cdot \\ \tanh\left(\frac{R^{-1}}{2\psi} \left(B(x) + \{Nx\}\right)^T \nabla\sigma_L^T(x) W_L \right) \end{matrix}, \sigma_j(x) \right\rangle \right) \cdot \sigma_j(x) \Bigg|\end{aligned}$$

$$\begin{aligned}&\leq \left| \sum_{k=1}^{L} w_k \sum_{j=L+1}^{\infty} \left\langle \nabla\sigma_k(x) A(x), \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right| \\ &+ \left| \sum_{j=L+1}^{\infty} \left\langle \left(x^T Q x\right) \cdot \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right| + \left| \left(\sum_{k=1}^{L} w_k \sum_{j=L+1}^{\infty} \left\langle M(\hat{u}), \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right) \right| \\ &+ \left| \left(\sum_{k=1}^{L} w_k \sum_{j=L+1}^{\infty} \left\langle \begin{matrix} \nabla\sigma_k(x) \left(B(x) + \{Nx\}\right)\psi \cdot \\ \tanh\left(\frac{R^{-1}}{2\psi} \left(B(x) + \{Nx\}\right)^T \nabla\sigma_L^T(x) W_L \right) \end{matrix}, \sigma_j(x) \right\rangle \cdot \sigma_j(x) \right) \right|\end{aligned}$$

Hence,

$$\begin{aligned}\left|\mathrm{HJB}(\hat{V}(x))\right| \leq D \cdot E(x) + F \cdot G(x) + D \cdot H(x) \\ + \left| \sum_{j=L+1}^{\infty} \left\langle \left(x^T Q x\right) \cdot \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right|\end{aligned}$$

where,

$$D = \max_{1 \leq k \leq L} |w_k|$$

$$E(x) = \sup_{x \in \Omega} \left| \sum_{k=1}^{L} \left(\sum_{j=L+1}^{\infty} \left\langle \nabla\sigma_k(x) \cdot A(x), \sigma_j(x) \right\rangle_\Omega \right) \cdot \sigma_j(x) \right|$$

$$F = 1$$

$$G(x) = \sup_{x \in \Omega} \left| \left(\sum_{k=1}^{L} \sum_{j=L+1}^{\infty} \left\langle M(\hat{u}), \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right) \right|$$

$$H(x) = \sup_{x \in \Omega} \left| \left(\sum_{k=1}^{L} \sum_{j=L+1}^{\infty} \left\langle \begin{matrix} \nabla\sigma_k(x) \left(B(x) + \{Nx\}\right)\psi \cdot \\ \tanh\left(\frac{R^{-1}}{2\psi} \left(B(x) + \{Nx\}\right)^T \nabla\sigma_L^T(x) W_L \right) \end{matrix}, \sigma_j(x) \right\rangle \cdot \sigma_j(x) \right) \right|$$

Suppose $\nabla\sigma_k(x)\cdot A(x)$, $M(\hat{u})$,

$$\psi\nabla\sigma_k(x)B(x)\tanh\left(\frac{R^{-1}}{2\psi}\left(B(x)+\{Nx\}\right)^T\nabla\sigma_L^T(x)W_L\right)$$

and x^TQx are in $L_2(\Omega)$, the orthogonally of the set $\{\sigma_j\}_1^\infty$ implies that $E(x)$ and second and third term on the right hand side can be made arbitrarily small by an appropriate choice of L.

Therefore, $D\cdot E(x)+F\cdot G(x)+D\cdot H(x)\to 0$

and $$\left|\sum_{j=L+1}^{\infty}\left\langle x^TQx,\sigma_j(x)\right\rangle_\Omega\cdot\sigma_j(x)\right|\to 0$$

So, $\left|\mathrm{HJB}(\hat{V}(x))\right|\to 0$ uniformly on Ω as L increases. Above lemma shows the existence of NN based HJB solution for the optimal control using cost functional $\int_0^\infty(x^TQx+M(u))dt$. Also approximate value function $\hat{V}(x)$ satisfies HJB equation (6) it means $e\to 0$. □

In the next theorem we will prove that, this optimal solution is valid for bilinear systems.

Theorem 2: If $\hat{u}$ is satisfies the HJB equation (9) then it is an optimal control of the bilinear system (1).

Proof: Lemma 1 shows the existence of NN based HJB solution.

So one can write equation (9) as

$$\mathrm{HJB}(\hat{V}(x))=x^TQx+M(\hat{u})+\hat{V}_x^T(A(x)+B(x)\hat{u})\approx 0 \tag{11}$$

Using above equation, approximate optimal control can be found similarly as in (8) by following equation

$$\begin{aligned}\hat{u}(x)&=-\psi\tanh\left(\frac{R^{-1}}{2\psi}\left(B(x)+\{Nx\}\right)^T\hat{V}_x\right)\\&=-\psi\tanh\left(\frac{R^{-1}}{2\psi}\left(B(x)+\{Nx\}\right)^T\nabla\sigma^T(x)W\right)\end{aligned} \tag{12}$$

Using (12) one can prove system's global stability, similar to theorem 1 by replacing $V(x)$ by $\hat{V}(x)$. □

Hence by using NN one can approximate the solution of the HJB equation. Next section is about the use of least squares method to find solution of HJB equation using NN.

IV. HJB Solution By Least-Squares Method

For the bilinear system, to solve equation (9), the method of weighted residuals is used [14].The unknown weights are determined by projecting the residual error e onto de/dW and setting the result to zero for $\forall x\in\Omega\subseteq\mathbb{R}^n$ using the inner product, i.e.

$$\langle de/dW,e\rangle=0 \tag{13}$$

where $\langle a,b\rangle=\int_\Omega abdx$ is a Lebesgue integral.

According to this method, by using definitions in equations (8) and (9), we can write (13) as

$$\begin{aligned}&\left\langle\begin{matrix}\nabla\sigma(x)(A(x)+B(x)\hat{u}+\{xN\}\hat{u}),\\\nabla\sigma(x)(A(x)+B(x)\hat{u}+\{xN\}\hat{u})\end{matrix}\right\rangle W\\&+\left\langle\begin{matrix}x^TQx+M(\hat{u}),\\\nabla\sigma(x)(A(x)+B(x)\hat{u}+\{xN\}\hat{u})\end{matrix}\right\rangle=0\end{aligned} \tag{14}$$

Hence weight updating law is

$$\begin{aligned}W=&-\left\langle\begin{matrix}\nabla\sigma(x)(A(x)+B(x)\hat{u}+\{xN\}\hat{u}),\\\nabla\sigma(x)(A(x)+B(x)\hat{u}+\{xN\}\hat{u})\end{matrix}\right\rangle^{-1}\\&\bullet\left\langle\begin{matrix}x^TQx+M(\hat{u}),\\\nabla\sigma(x)(A(x)+B(x)\hat{u}+\{xN\}\hat{u})\end{matrix}\right\rangle\end{aligned} \tag{15}$$

By solving this equation, one can find control using equation (12).

Using above weight one can find control which is the solution of constrained optimal control problem for the bilinear system. In the next section two numerical examples used to describe this proposed algorithm.

V. Simulation Experiments

To justify proposed algorithm we have implemented it on two bilinear systems.

(A) Consider a bilinear process for a continuously stirred tank reactor with an exothermic reaction defined by

$$\dot{x}_1=\frac{13}{6}x_1+\frac{5}{8}x_2-ux_1-\frac{u}{8} \quad\text{and}\quad \dot{x}_2=\frac{50}{3}x_1+\frac{8}{3}x_2$$

It is in the form of $\dot{x}=A(x)+B(x)u+\{xN\}u$

Here our aim is to find the optimal control law that will stabilize the system. For the above system we have to find a feedback control law $u=K(x)$ that minimizes

$$\int_0^\infty\left(x^TQx+M(u)\right)dt=\int_0^\infty(10x_1^{\,2}+10x_2^{\,2}+2\int_0^\Phi\tanh^{-1}(u)du)dt$$

where,

$$\Phi=-\tanh\left(\frac{R^{-1}}{2}\left(B(x)+\{Nx\}\right)^T\nabla\sigma^T(x)W\right),$$

$$Q=\begin{bmatrix}10&0\\0&10\end{bmatrix} \text{ and } R=1$$

Now the problem is converted into linear quadratic optimal control problem.

This problem can be solved by using equations (12) and (15). Here we have selected

$$\begin{aligned}\hat{V}(x)=&\,w_1x_1^2+w_2x_2^2+w_3x_1x_2+w_4x_1^4+w_5x_2^4\\&+w_6x_1^3x_2+w_7x_1^2x_2^2+w_8x_1x_2^3+w_9x_1^6+w_{10}x_2^6\end{aligned}$$

This is a NN with polynomial activation function and hence $\hat{V}(0)=0$. This is a power series NN of 10 activation functions containing power of the state variable of the system upto 6th order. The no. of weights required is chosen to guarantee the uniform convergence of the algorithm. NN based HJB solution can be found using least squares method. NN weights found from the (15) is

W =[0.9993;-0.0124;-0.0088;0.0182;0.0074;
0.0233;0.0013;0.0115;0.0096;-0.0006].

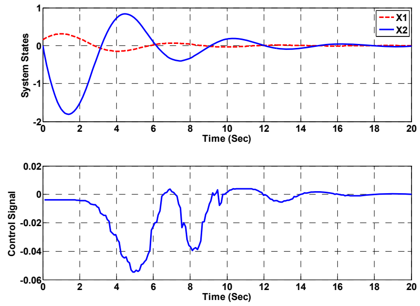

Fig. 1

Using these weights on can find optimal control from equation (12). Results are shown in Fig.1. System states converge to an equilibrium point. Variation in the control signal for this approach is shown in the same figure. Here control signal generated by NN remains bounded i.e. $|u| \le 1$. It shows that constrained optimal control converges to zero when system stabilized. It shows valid approximation of the solution of HJB equation with NN.

(B) Consider a paper making machine control problem described by the following bilinear model

$$\dot{x} = A(x) + B(x)u + \{xN\}u$$

where

$$A = \begin{bmatrix} -1.93 & 0 & 0 & 0 \\ 0.394 & -0.426 & 0 & 0 \\ 0 & 0 & -0.63 & 0 \\ 0.095 & -0.103 & 0.413 & -0.426 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.274 & 1.274 \\ 0 & 0 \\ 1.34 & -0.65 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.718 & -0.718 \\ 0 & 0 \end{bmatrix}$$

Here our aim is to find the optimal control law that will stabilize the system. For the above system we have to find a feedback control law $u = K(x)$ that minimizes

$$\int_0^\infty \left(x^T Q x + M(u)\right)dt = \int_0^\infty (x^T Q x + 2\int_0^\Phi \tanh^{-1}(u)du)dt$$

$$\Phi = -\tanh\left(\frac{R^{-1}}{2}\left(B(x) + \{Nx\}\right)^T \nabla\sigma^T(x)W\right),$$

where, $$Q = \begin{bmatrix} 1 & 0 & 0.13 & 0 \\ 0 & 1 & 0 & 0.09 \\ 0.13 & 0 & 0.1 & 0 \\ 0 & 0.09 & 0 & 0.2 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now the problem is converted into linear quadratic optimal control problem.

This problem can be solved by using equations (12) and (15). Here we have selected

$$\hat{V}(x) = w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 + w_4 x_4^2 + w_5 x_1 x_2 + w_6 x_1 x_3 + w_7 x_1 x_4 + w_8 x_2 x_3 + w_9 x_2 x_4 + w_{10} x_3 x_4$$

This is a NN with polynomial activation function and hence $\hat{V}(0) = 0$. This is a power series NN of 10 activation functions containing power of the state variable of the system upto 2nd order. The no. of weights required is chosen to guarantee the uniform convergence of the algorithm. NN based HJB solution can be found using least squares method. NN weights found from the (15) is

W =[0.9628;0.028;-0.0022;0.0318;-0.0553;0.0192;
0.0259;-0.2525;0.0574;0.0024];

Using these weights on can find optimal control from equation (12). Results are shown in Fig. 2. System states converge to an equilibrium point. Variation in the both control signals for this approach is shown in the same figure. Here control signal generated by NN remains bounded i.e. $|u| \le 1$. It shows that constrained optimal control converges to zero when system stabilized. It shows valid approximation of the solution of HJB equation with NN.

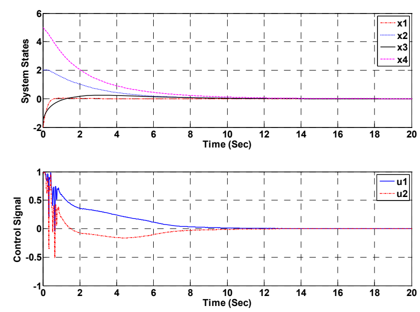

Fig. 2

VI. CONCLUSIONS

The contribution of this paper is a methodology for designing bounded controllers for bilinear systems. In the proposed frame work a constrained optimal control problem of bilinear systems solved by modifying a cost functional to account for constraint on the input. We have proposed NN based HJB solution for constrained optimal controller design. Modifications are done on the earlier approaches to handle constraint on the input for the bilinear systems. Least squares based method is used to find the solution of NN based HJB equation. We have achieved good results after modification of the performance function. It is also observed that control signal generated by NN remains bounded. Though there are many direct techniques available for the optimal control of bilinear systems, we have suggested an alternate approach using NN based HJB solution, for solving constrained optimal control problem. Furthermore, it is shown that the Lyapunov function guaranteeing stability is the solution to the HJB equation for the given system. This approach may be extended for other type of systems.

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