Neural Comput & Applic (2011) 20:91–103
DOI 10.1007/s00521-010-0441-1

ORIGINAL ARTICLE

Bounded robust control of nonlinear systems using neural network–based HJB solution

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Received: 17 June 2009 / Accepted: 27 August 2010 / Published online: 21 September 2010

Abstract In this paper, a Hamilton–Jacobi–Bellman (HJB) equation–based optimal control algorithm for robust controller design is proposed for nonlinear systems. The HJB equation is formulated using a suitable nonquadratic term in the performance functional to tackle constraints on the control input. Utilizing the direct method of Lyapunov stability, the controller is shown to be optimal with respect to a cost functional, which includes penalty on the control effort and the maximum bound on system uncertainty. The bounded controller requires the knowledge of the upper bound of system uncertainty. In the proposed algorithm, neural network is used to approximate the solution of HJB equation using least squares method. Proposed algorithm has been applied on the nonlinear system with matched and unmatched type system uncertainties and uncertainties in the input matrix. Necessary theoretical and simulation results are presented to validate proposed algorithm.

Keywords Robust control · HJB equation · Bounded control · System uncertainty · Lyapunov stability

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1 Introduction

In optimal control design, we need information about nominal system model. Optimal controller can be designed by solving well-known Hamilton–Jacobi–Bellman (HJB) equation. But it is very difficult to find exact solution of it, except for simple problems. Few literatures assumed the solution of HJB equation and formulated optimal control problem into standard linear quadratic regulator [1, 2]. Beard et al. [3, 4] have given an approximate solution by Galerkin's method and explored this method for underwater robotic vehicle control application. Neural network (NN)-based optimal control [5] uses a technique to approximate value function, which gives approximate solution of HJB equation. However, the design of a constrained optimal control using the solution of HJB equation is a challenging problem. Attempt is made in the literature to find constrained control law using nonquadratic performance functional [6–8]. All these HJB-based optimal controller designs need exact information about the system model and nominal model. However, all practical control systems have to be robust with respect to model uncertainty such as unknown or partially known time-varying process parameters and exogenous disturbances. So model uncertainty needs to be considered during the time of controller design process to avoid the deterioration of nominal closed-loop performance. In other words, we need to design robust feedback control law to tackle the system uncertainty.

Considerable efforts have been devoted to the development of robust control theory in recent years. It is important to quantitatively study performance degradations within a wide class of uncertainties and to design robust controllers to ensure closed-loop stability. Under special assumptions, the basic results in robust control of linear uncertain systems are reported in [9–11] and references therein. They proposed robust control design for time-varying model uncertainties based on Lyapunov stability theory. Robust control approach for nonlinear uncertain system with generalized matching condition is given in [12]. These concepts are shown to be manageable for a wide class of nonlinear uncertain systems, and design can be performed for time-invariant and time-varying systems. Although there exist many methods to address the robust control problems, it has been recognized that the Lyapunov concept and the Hamilton–Jacobi theory are the major analytic paradigms for designing and analysis of uncertain systems. Even though above-mentioned works provide systematic method for robust controller design, they do not in general lead to controllers that are optimal with respect to a meaningful cost. Lin [1, 2] proposed a method to find robust control using the solution of well-known HJB equation, which is difficult to solve in practice. In these papers, existence of the solution of HJB equation is assumed, without mentioning any procedure to find it. Algorithm is also developed to find robust control law without any *constraint (or bound)* on the control input. An alternative realistic approach is to obtain approximate solution of HJB equation using NN for feedback controller design for constrained input. However, constrained robust controller design using NN-based HJB solution has not been explored much in the literature.

In this paper, the robust control problem is formulated into an optimal control problem by properly choosing cost functional. The solution of optimal control problem becomes a solution to the robust control problem as well as that solution gives bounded control input. The cost functional is modified to account for system uncertainties and constraints on the input. Hence, it can be referred as a bounded robust-optimal control design approach. The main contribution is to realize this approach with necessary theoretical justifications and implementation of it through neural network. The least squares method is used to find tuning law of neural network, which is used to approximate the solution of HJB equation. In addition to this, we also explored proposed algorithm on the nonlinear uncertain systems (with matched and unmatched type system uncertainties and uncertainties in the input matrix) to ensure asymptotic stability of the closed-loop system. Moreover, it is shown that the Lyapunov function guaranteeing stability is a solution to the HJB equation for the nominal system. Convergence proof of the present work is supplemented by necessary theoretical and simulation results.

The paper is organized as follows: In Sect. 2, robust-optimal control framework has been described for system with matched and unmatched uncertainties and bounded input. In Sect. 3, NN-based HJB solution is used to find constrained robust-optimal control law. Stability issues with this formulation are also discussed, and the theoretical results are presented in the form of a lemma and theorems. Solution of NN-based HJB equation found by least squares method is in Sect. 4. Numerical examples are given in Sect. 5 for the validity of the approach explained in Sects. 3 and 4. Proposed work is concluded in Sect. 6.

2 Robust-optimal control framework

Consider a nonlinear system

$$\dot{x} = \bar{A}(x) + B(x)u$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is the control input. Each component of u is bounded by a positive constant λ

$$\text{i.e.,}\quad |u_i| \le \lambda \in \mathbb{R} \quad \text{where } i = 1, 2, \ldots, m \tag{1}$$

Suppose that the function $\bar{A}(x)$ is known only up to an additive perturbation that is bounded by a known function, and this perturbation is in the range of $B(x)$, i.e., if $\bar{A}(x)$ can be written as $\bar{A}(x) = A(x) + B(x)f(x)$ with unknown $f(x)$. The condition that unknown perturbation be in the range space of $B(x)$ is called the matching condition and can be incorporated by expressing the system as

$$\dot{x} = A(x) + B(x)u + B(x)f(x) \tag{2}$$

The function $B(x)f(x)$ models matched uncertainty in the system dynamics. The nominal model $A(x)$ and $B(x)$ are known with $A(0) = 0$ and $f(0) = 0$. This assumption ensures that origin is the equilibrium point of the system (2). It is also assumed that the function $f(x)$ is bounded by a known function, $f_{\max}(x)$:

$$\text{i.e.,}\quad \|f(x)\| \le f_{\max}(x). \tag{3}$$

In this paper, we seek a constrained optimal control that will compensate for the perturbation both for matched and for unmatched system uncertainties.

2.1 System with matched uncertainty

In this section, design of an optimal control law is proposed to ensure global asymptotic stability of the system (2).

(A) Robust control problem:

For the open-loop system (2), find a feedback control law $u = K(x)$ such that the closed-loop system is global asymptotically stable for all admissible uncertainties $f(x)$.

This problem can be formulated into an optimal control of the nominal system with appropriate cost functional.

(B) Optimal control problem:

For the nominal system,

$$\dot{x} = A(x) + B(x)u \tag{4}$$

finds a feedback control $u = K(x)$ that minimizes the cost functional

$$\int_0^\infty \left(\mu f_{\max}^2(x) + x^T x + M(u)\right) \mathrm{d}t, \tag{5}$$

where

$$M(u) = 2\int_0^u A\tan h^{-1}(v/\lambda)\mathrm{d}v = 2\lambda u \tan h^{-1}(u/\lambda) + \lambda^2 \ln(1 - u^2/\lambda^2) > 0 \tag{6}$$

is nonquadratic term expressing cost related to constrained input and $\mu > 0$ is design parameter.

In this paper, we addressed the following problems:

1. Solution of the robust control problem (A) and optimal control problem (B) are equivalent.
2. Solve the optimal control problem using NN. More specifically, HJB equation is solved using neural network.

To solve the optimal control problem, let $V(x_0) = \min_u \int_0^\infty \left(\mu f_{\max}^2(x) + x^T x + M(u)\right)\mathrm{d}t$ be the minimum cost of bringing the system (4) from initial condition x_0 to equilibrium point 0. The HJB equation gives

$$\min_u \left(\mu f_{\max}^2(x) + x^T x + M(u) + V_x^T (A(x) + B(x)u)\right) = 0 \tag{7}$$

where $V_x = \frac{\partial V(x)}{\partial x}$.

It is assumed that $V(x)$ is only a function of x. If $u = K(x)$ is the solution to the optimal control problem, then according to Bellman's optimality principle [13], it can be found by solving the following HJB equation:

$$\mathrm{HJB}(V(x)) = \mu f_{\max}^2(x) + x^T x + M(u) + V_x^T (A(x) + B(x)u) = 0 \tag{8}$$

The optimal control law is computed by solving

$$\frac{\partial \mathrm{HJB}(V(x))}{\partial u} = 0$$

$$\Rightarrow \frac{\partial M(u)}{\partial u} + V_x^T B(x) = 0$$

Using (6), the above equation can be written as

$$2\lambda \tan h^{-1}(u/\lambda) + V_x^T B(x) = 0 \tag{9}$$

$$\Rightarrow 2\lambda \tan h^{-1}(-u/\lambda) = V_x^T B(x)$$

$$\Rightarrow u = K(x) = -\lambda \tan h\left(\frac{1}{2\lambda} V_x^T B(x)\right) \tag{10}$$

With this basic introduction, following result is stated to show the equivalence of the solution of robust and the solution of optimal control problem.

Theorem 1 *Consider the nominal system* (4) *with the performance function* (5)*. Assume that there exits a function V(x), the solution of HJB equation* (8)*. Using this solution, bounded control law* (10) *ensures asymptotic closed-loop stability of uncertain nonlinear system* (2) *if following condition is satisfied:*

$$\mu f_{\max}^2(x) \geq V_x^T(x) B(x) f_{\max}(x) \tag{11}$$

Proof Here $u = K(x)$ is an optimal control law defined by (10), and $V(x)$ is the optimum solution of HJB equation (8). We now show that $u = K(x)$ is a solution to the robust control problem, i.e., the equilibrium point $x = 0$ of system (2) is asymptotically stable for all possible uncertainties $f(x)$. To do this, it is shown that $V(x)$ is a Lyapunov function. Clearly, $V(x)$ is a positive definite function, i.e.,

$$V(x) > 0, \quad x \neq 0 \quad \text{and} \quad V(0) = 0$$

$$\begin{aligned} \text{So,} \quad \dot{V}(x) &= (\partial V/\partial x)^T (\mathrm{d}x/\mathrm{d}t) \\ &= V_x^T(x)(A(x) + B(x)K(x) + B(x)f(x)) \\ &= V_x^T(x)(A(x) + B(x)K(x)) + V_x^T(x)B(x)f(x) \\ &\leq V_x^T(x)(A(x) + B(x)K(x)) + V_x^T(x)B(x)f_{\max}(x) \end{aligned}$$

Using (8) and (11), one can write

$$\begin{aligned} \dot{V}(x) &= -x^T x - M(u) - \left(\mu f_{\max}^2(x) - V_x^T(x)B(x)f_{\max}(x)\right) \\ &\leq -x^T x \\ &\leq 0 \end{aligned}$$

Thus, conditions for Lyapunov local stability theory are satisfied. □

(Note that the condition similar to (11) is satisfied for linear systems as well; proved in "Appendix 2"). Hence by knowing exact solution of HJB equation, one can find robust control law in the presence of matched uncertainties. In the next section, this approach extended to the system having uncertainties in the input matrix.

2.2 System having uncertainties in input matrix

To generalize the robust-optimal control approach for uncertainty in the input matrix, consider a nonlinear system defined by,

$$\dot{x} = A(x) + B(x)(u + h(x)u) + B(x)f(x) \tag{12}$$

where $B(x)h(x)$ models the uncertainties in the input matrix. Here, our goal is to seek an optimal control that will compensate for the perturbations $h(x)$ and $f(x)$.

It is assumed that uncertainties $h(x)$ and $f(x)$ satisfies the following conditions:

(i)

$$h(x) \geq 0. \tag{13}$$

(ii) Uncertainty $f(x)$ bounded by a known function $f_{\max}(x)$ as defined in (3), i.e., $\|f(x)\| \leq f_{\max}(x)$

Note that the above-mentioned conditions arise in some practical situation, which is elaborated in this paper by considering a robot manipulator example. It is required to design an optimal control law $u = K(x)$ to ensure the closed-loop stability of (12). This problem can be formulated into same optimal control problem (B) as discussed in Sect. 2.1. The control law (10) can be computed by solving HJB equation (8). Equivalence of the solution of optimal control problem and that of robust control problem can be proved similarly as Theorem 1 using additional condition (13). The details are not given here. In the next section, similar approach explored to the system having unmatched uncertainties.

2.3 System having unmatched uncertainties

To generalize the robust-optimal control approach for handling the unmatched uncertainty, consider a nonlinear system

$$\dot{x} = A(x) + B(x)u + C(x)f(x) \tag{14}$$

where $C(x)$ is a matrix of dimension $n \times q$ and $C(x) \neq B(x)$. It is assumed that $f(0) = 0$ and $A(0) = 0$, so that $x = 0$ is an equilibrium point of (14). It is required to find a bounded optimal control law, which ensures the closed-loop stability of the system (14) for unmatched uncertainties $f(x)$. Similar to Sect. 2.1, one can formulate this problem as

(A) Robust control problem:

For the open-loop system (14), find a feedback control law $u = K(x)$ such that the closed-loop system is global asymptotically stable for all uncertainties $f(x)$ satisfying the following conditions:

(i) $f(x)$ is bounded as defined in (3).
i.e. $\|f(x)\| \leq f_{\max}(x)$

(ii) There exists a non-negative function $g_{\max}(x) \geq 0$ such that

$$\left\|B(x)^{+}C(x)f(x)\right\| \leq g_{\max}(x) \tag{15}$$

where $^{+}$ denotes the (Moore–Penrose) pseudo-inverse.

Decomposition of the uncertainty term $C(x)f(x)$ can be done as the sum of matched and an unmatched component by projecting $C(x)f(x)$ onto the range of $B(x)$. It can be written as

$$\begin{aligned} C(x)f(x) &= B(x)B(x)^{+}C(x)f(x) \\ &\quad + (I - B(x)B(x)^{+})C(x)f(x) \end{aligned} \tag{16}$$

The robust control problem

(A) can be solved using following optimal control problem:

(B) Optimal control problem:

Define an auxiliary system having unmatched uncertainty component of (16):

$$\dot{x} = A(x) + B(x)u + (I - B(x)B(x)^{+})C(x)v \tag{17}$$

where (u, v) is the control input. Find a feedback control (u, v) that minimizes the performance cost

$$\int_{0}^{\infty} \left(\mu g_{\max}^{2}(x) + \rho^{2} f_{\max}^{2}(x) + \beta^{2}\|x\|^{2} + M(u) + \rho^{2}\|v\|^{2}\right) \mathrm{d}t \tag{18}$$

where ρ, β and μ are some positive constants that serve as design parameters. Note that the optimal control of system (17) has two components: $u \in \mathbb{R}^{>}$ and $v \in \mathbb{R}^{''}$. It is very difficult to design a control law for system with unmatched uncertainties. In support to control u, an augmented control v added in the system to tackle the unmatched component of uncertainties defined by (16). However, in the actual system (14), only u component is used. An augmented control v plays important role for proving asymptotic closed-loop stability of the system (14). It will be discussed in Theorem 2.

To solve optimal control problem, let $V(x_0) = \min_{u,v} \int_0^{\infty} \left(\mu g_{\max}^{2}(x) + \rho^{2} f_{\max}^{2}(x) + \beta^{2}\|x\|^{2} + M(u) + \rho^{2}\|v\|^{2}\right) \mathrm{d}t$ to be the minimum cost of bringing the system (17) from initial condition x_0 to equilibrium point 0. The HJB equation gives us

$$\begin{aligned} &\min_{u,v}(\mu g_{\max}^{2}(x) + \rho^{2} f_{\max}^{2}(x) + \beta^{2}\|x\|^{2} + M(u) + \rho^{2}\|v\|^{2} \\ &\quad + V_x^T(A(x) + B(x)u + (I - B(x)B^{+}(x))C(x)v)) = 0 \end{aligned}$$

where $V_x = \frac{\partial V(x)}{\partial x}$. Since it is assumed that $V(x)$ is only a function of x, according to Bellman's optimality principle [13], the optimal cost is obtained by solving the following HJB equation:

$$\begin{aligned} \mathrm{HJB}(V(x)) = \mu g_{\max}^{2}(x) &+ \rho^{2} f_{\max}^{2}(x) + \beta^{2}\|x\|^{2} + M(u) \\ &+ \rho^{2}\|v\|^{2} + V_x^T(A(x) + B(x)u \\ &+ (I - B(x)B^{+}(x))C(x)v) = 0 \end{aligned} \tag{19}$$

which gives the optimal control

$$u = K(x) = -\lambda \tanh\left(\frac{1}{2\lambda} V_x^T B(x)\right) \tag{20}$$

and

$$v(x) = -\frac{1}{2\rho^2} V_x^T (I - BB^+) C \quad (21)$$

Following theorem proves that the control law (20) is the solution of the robust control problem.

Theorem 2 *Consider the nominal system (17) with performance function (18). Assume that there exists a function V(x), the solution of HJB equation (19) by properly choosing ρ and β. Using this solution, control law (20) ensures asymptotic closed-loop stability of uncertain nonlinear system (14) for some β^* such that $|\beta^*| < |\beta|$, if following conditions are satisfied:*

$$2\rho^2 \|v\|^2 \le \beta^{*2} \|x\|^2, \quad \forall x \in \mathbb{R}^n \quad (22)$$

$$\mu g_{\max}^2 \ge V_x^T B(x) g_{\max}(x), \quad \forall x \in \mathbb{R}^n \quad (23)$$

Proof It is proved here that using control law defined in (20) and (21), the system (14) remains asymptotically stable for all possible *f*(*x*). To do this, we show that *V*(*x*) is a Lyapunov function.

Clearly, $V(0) = 0$ and $V(x) > 0$ for $\forall x \neq 0$.

The time derivative of $V(x)$ is shown to be negative definite.

$$\begin{aligned}
\dot{V}(x) &= (\partial V/\partial x)^T (\mathrm{d}x/\mathrm{d}t) \\
&= V_x^T (A(x) + B(x)u + C(x)f(x)) \\
&= V_x^T (A(x) + B(x)u + (I - B(x)B^+(x))C(x)v) \\
&\quad + B(x)B^+(x)C(x)f(x) + (I - B(x)B^+(x))C(x)(f(x) - v)) \\
&= V_x^T (A(x) + B(x)u + (I - B(x)B^+(x))C(x)v)) \\
&\quad + V_x^T B(x)B^+(x)C(x)f(x) \\
&\quad + V_x^T (I - B(x)B^+(x))C(x)(f(x) - v)
\end{aligned}$$

Using (19) and (20), we have

$$\begin{aligned}
\dot{V}(x) = &- \mu g_{\max}^2(x) - \rho^2 f_{\max}^2(x) - \beta^2 \|x\|^2 - M(u) - \rho^2 \|v\|^2 \\
&+ V_x^T B(x)B^+(x)C(x)f(x) - 2\rho^2 v^T (f(x) - v)
\end{aligned}$$

Since

$$-2\rho^2 v^T f(x) \le \rho^2 \left(\|v\|^2 + \|f(x)\|^2 \right)$$

we can write

$$\begin{aligned}
\dot{V}(x) \le &- M(u) - \left(\mu g_{\max}^2 - V_x^T B(x) g_{\max}(x)\right) \\
&- \rho^2 \left(f_{\max}^2 - \|f(x)\|^2\right) - \|u + B^+(x)C(x)f(x)\|^2 \\
&+ 2\rho^2 \|v\|^2 - \beta^2 \|x\|^2 \\
\le &- M(u) - \left(\mu g_{\max}^2 - V_x^T B(x) g_{\max}(x)\right) \\
&- \rho^2 \left(f_{\max}^2 - \|f(x)\|^2\right) - \|u + B^+(x)C(x)f(x)\|^2 \\
&+ 2\rho^2 \|v\|^2 - \beta^{*2} \|x\|^2 - \left(\beta^2 - \beta^{*2}\right) \|x\|^2
\end{aligned}$$

Using conditions (22) and (23),

$$\dot{V}(x) \le -\left(\beta^2 - \beta^{*2}\right) \|x\|^2 < 0$$

Thus, the conditions of the Lyapunov stability theorem are satisfied. □

Theorems 1 and 2 are valid if we know the exact solution of HJB equation, which is difficult problem. In the next section, NN is used to approximate value function *V*, which is the solution of HJB equation.

3 NN-based robust-optimal control

In this section, we used NN to find approximate solution of HJB equation, which is utilized to find robust-optimal control. It is well known that a NN can be used to approximate the function on a prescribed compact set [14]. It can be used to approximate a nonlinear mapping. Let $\mathbb{R}$ denote the real numbers. Given $x_k \in \mathbb{R}$, define $x = [x_0, x_1, \ldots, x_n]^T$, $y = [y_0, y_1, \ldots, y_m]^T$ and weight matrices $W = [w_1, w_2, \ldots, w_L]^T$. Then, the ideal NN output can be expressed as $y = W^T \sigma(x)$ with the vector of NN activation functions $\sigma(x) = [\sigma_1(x), \sigma_2(x), \ldots, \sigma_L(x)]^T$. It is assumed to be orthonormal and satisfy the NN approximation property [8]. With this background, we proposed the robust-optimal control framework based on NN in the next section.

3.1 System with matched uncertainties

Let the structure of NN-based approximate value function can be defined as

$$\hat{V}(x) = \sum_{j=1}^{L} w_j \sigma_j(x) = W^T \sigma(x) \quad (24)$$

where $W = [w_1, w_2, \ldots, w_L]^T$ is the set of NN weights. $\sigma(x)$ is selected such that $\hat{V}(0) = 0$ and $\hat{V}(x) > 0 \ \forall x \neq 0$. So

$$\hat{V}_x(x) = \frac{\partial \hat{V}}{\partial x} = W^T \nabla \sigma(x) \quad (25)$$

HJB equation with this assumption is as follows:

$$\begin{aligned}
\mathrm{HJB}(\hat{V}(x)) &= \mu f_{\max}^2(x) + x^T x + M(\hat{u}) + \hat{V}_x^T (A(x) + B(x)\hat{u}) \\
&= e
\end{aligned} \quad (26)$$

Here NN is used to approximate value function *V*(*x*). Approximation error is represented by *e*. If *e* is negligible, then (26) becomes similar to HJB equation (8). We have introduced a lemma to show the existence of NN-based HJB solution for optimal control using modified performance functional. Theorem 3 shows the relationship between the robust control and the optimal control for NN-based HJB solution.

Lemma 1 *Let* $\hat{V}(x) = \sum_{j=1}^{L} w_j\sigma_j(x)$ *satisfy* $\langle \mathrm{HJB}(\hat{V}(x)), \sigma(x)\rangle_\Omega = 0$ *and* $\langle \hat{V}(x), \sigma(x)\rangle_\Omega = 0$ *on a compact set* $\Omega \subset \mathbb{R}^n$, *and let* $V(x) = \sum_{j=1}^{\infty} c_j\sigma_j(x)$ *and* $C = [c_1, c_2, \ldots, c_L]^T$ *satisfy HJB* $(V(x)) = 0$. *If* Ω *is compact,* $(\mu f_{\max}^2(x) + x^T x)$ *is continuous on* Ω *and is in the space span* $\{\sigma_j\}_1^\infty$, *and if the coefficients* $|w_j|$ *are uniformly bounded for all L, then* $|\mathrm{HJB}(\hat{V}(x))| \to 0$ *uniformly on* Ω *as* L increases.

Proof Khalaf et al. [7, 15] proposed theorem for the existence of NN-based HJB solution for the optimal control problem. The existence of NN-based HJB solution for optimal control using modified performance functional can be proved in the same line of it. □

Lemma 1 shows the existence of NN-based HJB solution. So, (26) can be written as

$$\mathrm{HJB}(\hat{V}(x)) = \mu f_{\max}^2(x) + x^T x + M(\hat{u}) + \hat{V}_x^T(A(x) + B(x)\hat{u}) \approx 0 \tag{27}$$

The optimal control law can be found by taking derivative of (27) w.r.t. $\hat{u}$. It can be found as

$$\hat{u}(x) = -\lambda \tanh\left(\frac{1}{2\lambda}B(x)^T\hat{V}_x\right) = -\lambda \tanh\left(\frac{1}{2\lambda}B(x)^T W\nabla\sigma^T(x)\right) \tag{28}$$

In the next theorem, equivalence of the NN-based solution of optimal control problem and robust control problem is proved.

Theorem 3 *Assume that the NN-based HJB solution to the optimal control problem exists. Then, control law defined by* (28) *ensures closed-loop asymptotic stability of nonlinear uncertain system* (2) *if following condition satisfied:*

$$\mu f_{\max}^2(x) \geq \hat{V}_x^T B(x) f_{\max}(x)$$

Proof Here $\hat{u}(x)$ is an optimal control law defined by (28), and $\hat{V}(x)$ is the solution of the HJB equation (27). We now show that with this control, the system remains asymptotically stable for all possible $f(x)$. Using definition (24) and from the selection of $\sigma(x)$, $\hat{V}(0) = 0$ and $\hat{V}(x) > 0 \quad \forall x \neq 0$. Also $\dot{\hat{V}}(x) = \frac{d\hat{V}}{dt} < 0$ for $x \neq 0$, can be proved similarly as Theorem 1 by replacing $V(x)$ by $\hat{V}(x)$. □

From the above theorem, it can be proved that instead of solving the robust control problem, one can solve the optimal control problem. It is shown that this procedure always leads to a robust control that stabilizes system having matched uncertainty. Note that controller formulation used for the matched uncertainties part can be extended for additional uncertainties in the input matrix as per description in Sect. 2.2. For this case, one can have same HJB equation (27) and control law (28). So, closed-loop stability can be proved similar to Theorem 3.

3.2 System having unmatched uncertainties

As described in Sect. 3.1, one can find robust-optimal control for the system with matched uncertainties using NN-HJB approach. In this section, similar framework is extended for the system having unmatched uncertainties. Using the similar structure of NN-based approximate value function, the HJB equation can be written as

$$\mathrm{HJB}(\hat{V}(x)) = \mu g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(\hat{u}) + \rho^2\|\hat{v}\|^2 + \hat{V}_x^T(A(x) + B(x)\hat{u} + (I - B(x)B^+(x))C(x)\hat{v}) = e \tag{29}$$

Here NN is used to approximate value function $V(x)$. Approximation error is represented by e. If e is negligible, then (29) becomes similar to (19). One can show the existence of NN-based HJB solution for the above performance functional by the similar kind of proof as in Lemma 1.

As $e \to 0$, we can write (29) as

$$\mathrm{HJB}(\hat{V}(x)) = \mu g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(\hat{u}) + \rho^2\|\hat{v}\|^2 + \hat{V}_x^T(A(x) + B(x)\hat{u} + (I - B(x)B^+(x))C(x)\hat{v}) \approx 0 \tag{30}$$

Approximate optimal controls can be found by taking derivative of (30) w.r.t. $\hat{u}$ and $\hat{v}$:

$$\hat{u}(x) = -A \tanh\left(\frac{1}{2A}B(x)^T\hat{V}_x\right) = -A \tanh\left(\frac{1}{2A}B(x)^T W\nabla\sigma^T(x)\right) \tag{31}$$

and

$$\hat{v}(x) = -\frac{1}{2\rho^2}(I - BB^+)^T C^T W\nabla\sigma^T(x) \tag{32}$$

The relationship between robust control and optimal control for NN-based HJB solution can be defined similar to Theorems 2 and 3 for the unmatched uncertainty case. It can be observed for unmatched uncertainty case that the robust control problem can be solved by solving corresponding optimal control problem. We have shown that this procedure always leads to a control law that stabilizes the uncertain system. Next section is about the utilization of the least squares method for finding a HJB solution.

4 HJB solution by least square method

Method of weighted residuals [7, 15] was explored for optimal control problem. The unknown weights are determined by projecting the residual error e onto $\mathrm{d}e/\mathrm{d}W$ and setting the result to zero using the inner product, i.e.,

$$\langle \mathrm{d}e/\mathrm{d}W, e\rangle = 0 \quad \forall x \in \Omega \subseteq \mathbb{R}^n \tag{33}$$

where $\langle a, b\rangle = \int_\Omega ab\mathrm{d}x$ is a Lebesgue integral.

This method can be applied to solve robust-optimal control problem for the system having matched uncertainties. According to this method, by using definitions (25) and (26), we can write (33) as

$$\langle \nabla\sigma(x)(A(x)+B(x)\hat{u}), \nabla\sigma(x)(A(x)+B(x)\hat{u})\rangle W + \left\langle \mu f_{\max}^2(x) + x^T x + M(\hat{u}), \nabla\sigma(x)(A(x)+B(x)\hat{u})\right\rangle = 0 \tag{34}$$

Hence, weight updating law for the matched uncertainty case is

$$W = -\langle \nabla\sigma(x)(A(x)+B(x)\hat{u}), \nabla\sigma(x)(A(x)+B(x)\hat{u})\rangle^{-1} \cdot \left\langle \mu f_{\max}^2(x) + x^T x + M(\hat{u}), \nabla\sigma(x)(A(x)+B(x)\hat{u})\right\rangle \tag{35}$$

By solving this equation, one can find control law using (28), which is the solution of robust control problem having matched uncertainties. Similar HJB solution can be found for unmatched uncertainties. The inner product in (33) can be written using (25) and (29) as

$$\langle \nabla\sigma(x)(A(x)+B(x)\hat{u}), \nabla\sigma(x)(A(x)+B(x)\hat{u})\rangle W + \left\langle \mu g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(\hat{u}) + \rho^2\|\hat{v}\|^2, \nabla\sigma(x)(A(x)+B(x)\hat{u})\right\rangle = 0 \tag{36}$$

Hence, weight updating law is

$$W = -\langle \nabla\sigma(x)(A(x)+B(x)\hat{u}), \nabla\sigma(x)(A(x)+B(x)\hat{u})\rangle^{-1} \cdot \left\langle \mu g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(\hat{u}) + \rho^2\|\hat{v}\|^2, \nabla\sigma(x)(A(x)+B(x)\hat{u})\right\rangle \tag{37}$$

By solving this equation, one can find control law using (31), which is the solution of robust control problem for the systems having unmatched uncertainties. In the next section, proposed algorithm has been described in terms of appropriate steps.

4.1 Algorithm

Proposed algorithm is described in the following steps:

(i) Initialize the iteration index and number of mesh points p and select an appropriate structure of $\sigma(x)$.

(ii) Randomly initialize system states in the state space $\Omega \subseteq \mathbb{R}^n$ for p mesh points. For the nominal system, find a stabilizing optimal control law.

(iii) Using control law of step (ii) and (31) (for matched uncertainties case) or (37) (for unmatched uncertainties case), find weights of the neural network

(iv) Increase iteration index by one.

(v) Repeat steps (ii)–(iv) for sufficient amount of iterations.

(vi) Using the weights of the last iterations, find control law given by (28) (for matched uncertainties case) or (31) (for unmatched uncertainties case).

It is to be noted that the number of mesh points should be selected sufficiently large such that it can cover all type of variations of the system states in Ω.

This is an offline algorithm run prior to obtain a neural network–based bounded state feedback controller. After running an offline algorithm, it is necessary to verify that NN-based approximate solution of HJB equation satisfies the Lyapunov function property, i.e., $\hat{V}(x) > 0$ and $\dot{\hat{V}}(x) \leq 0$ $\forall x \neq 0$. If it not, then one has to repeat the algorithm. If it satisfy, then controller designed using $\hat{V}(x)$, can be proved to be a robust controller as explained in Theorem 3. However, it is also necessary to verify condition (11) for the approximate value function. If condition (11) is not satisfied, then one has to repeat algorithm steps with modification of the number of mesh points or the order of basis function $\sigma(x)$. In the next section, simulation carried out on three uncertain systems to validate proposed algorithm.

5 Simulation experiments

5.1 Matched uncertainties case

In this section, we have explored proposed algorithm on a second-order and a fourth-order nonlinear uncertain systems.

(i) Consider the nonlinear system

$$\dot{x}_1 = x_1 + x_2 - x_1\left(x_1^2 + x_2^2\right)$$
$$\dot{x}_2 = -x_1 + x_2 - x_1\left(x_1^2 + x_2^2\right) + u + px_1\sin x_2$$

where p is the unknown parameter and control input is bounded by $|u| \leq 1$. For simplicity, let assume that $p \in [-1, 1]$. It is in the matched uncertainty form,

i.e. $\dot{x} = A(x) + B(x)u + B(x)f(x)$ with $f(x) = px_1\sin x_2$.

Clearly, $|f(x)| \leq |x_1| = f_{\max}(x)$. $\mu = 10$ has been selected for the simulation purpose.

Here, our aim is to find the robust control law that will stabilize the system for all possible p. This problem can be formulated into the following optimal control problem:

For the nominal system,

$$\dot{x}_1 = x_1 + x_2 - x_1\left(x_1^2 + x_1^2\right);$$
$$\dot{x}_2 = -x_1 + x_2 - x_1\left(x_1^2 + x_1^2\right) + u$$

we have to find a feedback control law $u = K(x)$ that minimizes

$$\int_0^\infty \left(10f_{\max}^2(x) + x^T x + M(u)\right)\mathrm{d}t = \int_0^\infty \left(11x_1^2 + x_2^2 + 2\int_0^\Phi \tanh^{-1}(u)\mathrm{d}u\right)\mathrm{d}t$$

where $\Phi = -\tanh\left(\frac{1}{2}W^T\nabla\sigma(x)B(x)\right)$. This problem can be solved by using (25) and (35). The scalar parameter $p = 1$ has been selected for the purpose of simulating the plant. Here, we have selected

$$\begin{aligned}\hat{V}(x) = {} & w_1x_1^2 + w_2x_2^2 + w_3x_1x_2 + w_4x_1^4 + w_5x_2^4 + w_6x_1^3x_2 \\ & + w_7x_1^2x_2^2 + w_8x_1x_2^3 + w_9x_1^6 \\ & + w_{10}x_2^6 + w_{11}x_1^5x_2 + w_{12}x_1^4x_2^2 + w_{13}x_1^3x_2^3 \\ & + w_{14}x_1^2x_2^4 + w_{15}x_1x_2^5 + w_{16}x_1^8 + w_{17}x_2^8 \\ & + w_{18}x_1^7x_2 + w_{19}x_1^6x_2^2 + w_{20}x_1^5x_2^3 + w_{21}x_1^4x_2^4 \\ & + w_{22}x_1^3x_2^5 + w_{23}x_1^2x_2^6 + w_{24}x_1x_2^7\end{aligned}$$

This is a NN with polynomial activation function and hence $\hat{V}(0) = 0$. This is a power series NN of 24 activation functions containing power of the state variable of the system up to eighth order. Selecting the NN structure for approximating $V(x)$ is usually a natural choice guided by engineering experience and intuition. It requires trial-and-error procedure. However, in our case, the polynomial structured NN suggested in [7, 16] has worked. So, we have considered it. The number of neurons required is chosen to guarantee the uniform convergence of the algorithm. Neurons with eight-order power of the states variables were selected because for neurons with 6th power of the states, the algorithm did not converge. Higher-order power terms were producing similar results without much improvement. Hence to avoid computational complexity, we have taken activation function up to eighth order. The activation functions for the neural network selected in this paper satisfy the properties of the activation function discussed in [7]. The number of mesh points has been selected to run an offline algorithm presented in Sect. 4.1 is 1,500. NN based HJB solution can be found using least squares method as described in the algorithm. It gives $W = [30.0463; 10.7230; 21.0872; 33.8288; 1.3228; 41.3044; 28.2090; 2.5712; 19.8882; 2.368940.8138; 41.7514; 10.0255; 4.7980; 4.9658; 3.5467; 0.4724; 13.5306; 20.6493; 10.3097; 0.0601; 4.4958; 3.5604; 0.6157]$. The robust control law can be found using (28). It can be observed from Fig. 1a that all the system states converge to the equilibrium point. Control signal remains bounded, i.e., $|u| \leq 1$, as shown in Fig. 1b. It can also observed from the Fig. 1c, d that the Lyapunov function $\hat{V}(x) \geq 0$ and $\dot{\hat{V}}(x) \leq 0$ for all the values of x. Condition (11) of Theorem 1 in the stability proof of robust controller design has been verified and shown in Fig. 2. It shows valid approximation of the solution of HJB equation with neural network. It also shows that constrained robust-optimal control input converges to zero when system is stabilized.

(ii) Robot controller design

In this section, we will illustrate proposed approach by an example of two-joint SCARA type robot. The configuration of the robot manipulator and its parameters are shown in Fig. 3.

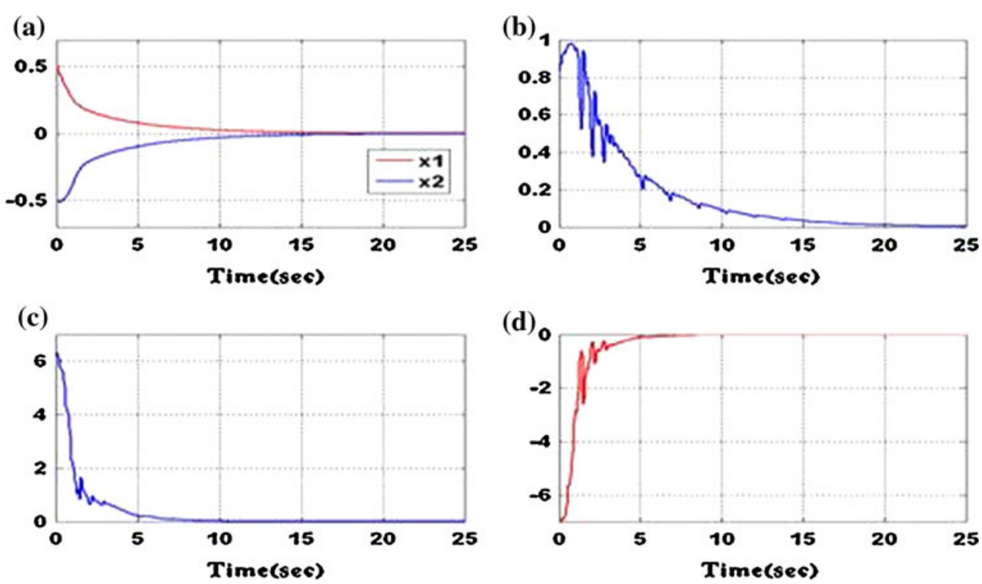

Fig. 1 **a** System states versus time, **b** variation of control input, **c** Lyapunov function and **d** derivative of Lyapunov function

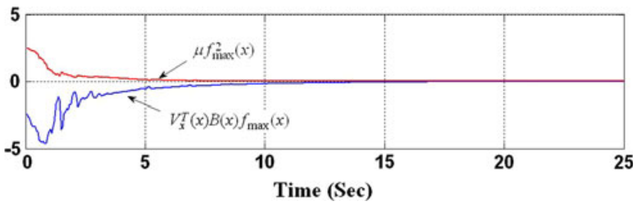

Fig. 2 Verification of condition (11)

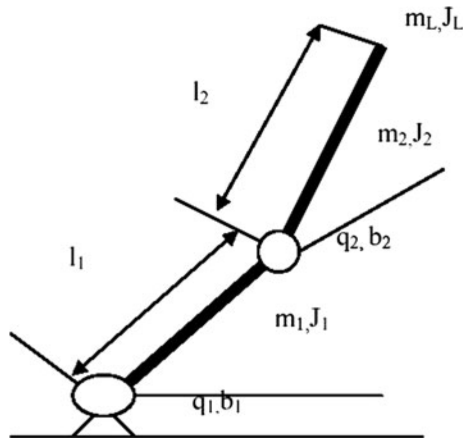

Fig. 3 SCARA Robot

The dynamics of robot manipulator [17] is given by

$$M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q) = \tau$$

where q consists of joint variables, τ is the generalized forces, $M(q)$ the inertia matrix, $V(q,\dot{q})$ the centripetal vector, $G(q)$ the gravity vector and $F(\dot{q})$ the friction vector. For simplicity, we denote $N(q,\dot{q}) = V(q,\dot{q}) + F(\dot{q}) + G(q)$. There are uncertainties in $M(q)$ and $N(q,\dot{q})$ due to, say, unknown load on the manipulator and unmodeled frictions. All the terms of system dynamics are explained in "Appendix 1". With those details, state equation of the system can be written as

$$\dot{x} = Ax + B(u + h(x)u) + Bf(x)$$

where

$$[x_1\ x_2\ x_3\ x_4]^T = [q_1\ q_2\ \dot{q}_1\ \dot{q}_2]^T; \quad A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ I \end{bmatrix};$$
$$h(x) = M(x_1)^{-1}M_0(x_1) - I \geq 0$$

Although for the above $f(x)$, $\|f(x)\|^2$ may not be quadratically bounded. In many cases, we can find the largest feasible region of x and determine a quadratic bound $\|f(x)\|^2$. Assume such a quadratic bound is given by $f(x)^T f(x) \leq x^T Px$ for some positive semidefinite matrix P. $\mu = 1$ and

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 218 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$$

have been selected for simulation purpose. Here, our aim is to find the robust control law that will stabilize the system defined in the form of (12) for all possible $h(x)$ and $f(x)$. This problem can be solved by using (25) and (35). Here we have selected

$$\hat{V}(x) = w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + w_4x_4^2 + w_5x_1x_2 + w_6x_1x_3 + w_7x_1x_4 + w_8x_2x_3 + w_9x_2x_4 + w_{10}x_3x_4$$

This is a NN with polynomial activation function and hence $\hat{V}(0) = 0$. It is a power series NN of 10 activation functions containing powers up to second order of the state variables of the system. Neurons with second-order power of the state variables were selected because for neurons with 1st power of the states, the algorithm did not converge. Higher-order power terms were producing similar results without much improvement. Hence to avoid computational complexity, we have taken activation function up to second order. The number of mesh points has been selected to run an offline algorithm presented in Sect. 4.1 is 1,500. In this example, we found

$$\mathbf{W} = [2.0641\ 0.6862\ 2.7643\ 3.3230\ 0.4214\ 3.1927\ 0.0178\ -0.0193\ 5.8515\ 2.6898]$$

The robust control law can be found using (28). It remains bounded, i.e., $|u_1| \leq 5$ and $|u_2| \leq 2$. In the present example of a robot manipulator, system dynamics modeled as a system having matched uncertainty with uncertainty in the input matrix. NN weights have been tuned by nominal system. Variations in the system states and control signals are shown in Figs. 5a and 4b. NN-based Lyapunav function (Fig. 4c) and its time derivatives (Fig. 4d) shows stabilized

$$f(x) = M(q)^{-1}(N_0(q,\dot{q}) - N(q,\dot{q})) = \begin{bmatrix} 846 + 60\varepsilon^2 + 1000\varepsilon + 172\cos q_2 + 960\varepsilon\cos q_2 & 51 + 60\varepsilon^2 + 360\varepsilon + 86\cos q_2 \\ 51 + 60\varepsilon^2 + 360\varepsilon + 86\cos q_2 & 51 + 60\varepsilon^2 + 360\varepsilon \end{bmatrix}^{-1} \cdot \begin{bmatrix} 480(1-\varepsilon)(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)\sin q_2 + 0.2q_1 \\ 480(1-\varepsilon)\dot{q}_1^2\sin q_2 + 0.5q_2 \end{bmatrix}$$

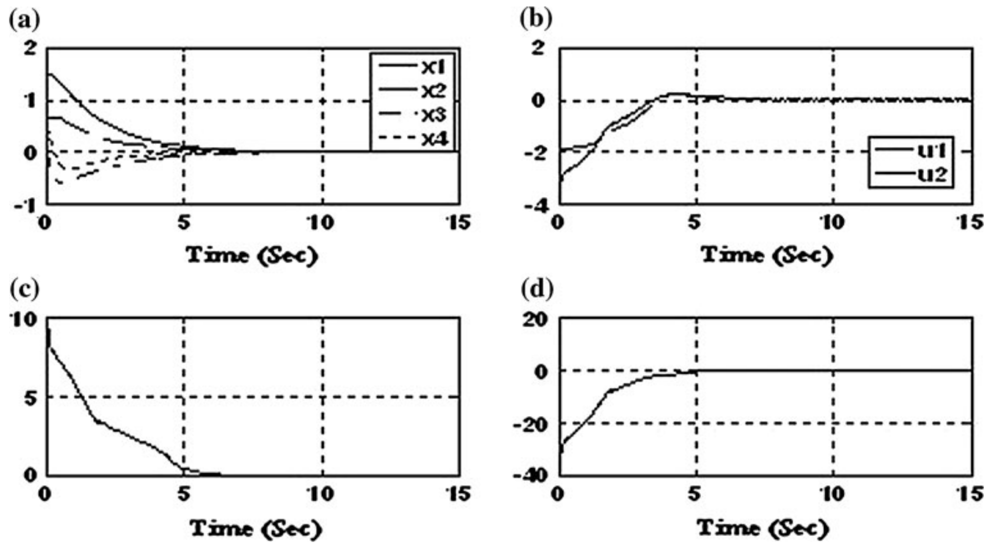

Fig. 4 **a** System states versus time, **b** variation of control input, **c** Lyapunov function and **d** derivative of Lyapunov function

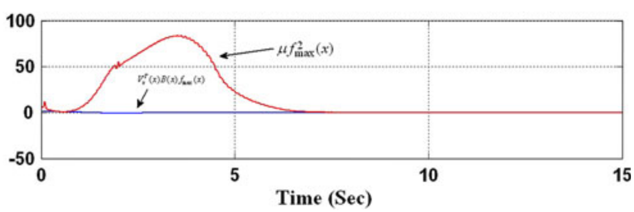

Fig. 5 Verification of condition (11)

control. Verification of the condition (11) of Theorem 1 can be observed from the Fig. 5. Convergence of the system state to the equilibrium point validates proposed algorithm.

5.2 Unmatched uncertainties case

Let the nonlinear system defined by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u + \begin{bmatrix}1\\ 0\end{bmatrix}f(x_1, x_2)$$

where,

$$f(x_1, x_2) = 5p_1x_1\cos\left(\frac{1}{x_2 + p_2}\right) + 5p_3x_2\sin(p_4x_1x_2)$$

with

$p_1(t) \in [-0.2, 0.01]$, $p_2(t) \in [-100, 100]$, $p_3(t) \in [-0.05, 0.05]$

and $p_4(t) \in [-100, 0]$ are time-varying uncertainties, and control input is bounded by $|u| \le 50$. It is in the unmatched uncertainty form, i.e., $\dot{x} = A(x) + B(x)K(x) + C(x)f(x)$.

Therefore, $\|f(x_1, x_2)\|^2 \le x_1^2 + x_2^2 = g_{\max}^2(x)$ and $\|B^+Cf(x_1, x_2)\|^2 = 0 = f_{\max}^2(x)$

Also, $B^+ = (B^TB)^{-1}B^T = B^T = [0\,1]$ and

$$(I - BB^+)C = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0.2\\ 0\end{bmatrix} = \begin{bmatrix}0.2\\ 0\end{bmatrix}$$

$\mu = 4$ and $\rho = \beta = 1$ are selected for the purpose of the simulation. As per description in Sects. 2 and 3, the corresponding optimal control problem is as follows: For the system,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u + \begin{bmatrix}0.2\\ 0\end{bmatrix}v$$

finds a feedback control law that minimizes the cost

$$\int_0^\infty \left(\mu g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + \rho^2\|v\|^2 + M(u)\right)\mathrm{d}t$$
$$= \int_0^\infty \left(5x_1^2 + 5x_2^2 + v^Tv + 100\int_0^\Phi \tanh^{-1}(u/50)\mathrm{d}u\right)\mathrm{d}t$$

where $\Phi = -50\tanh\left(\frac{1}{100}W^T\nabla\ \sigma(x)B(x)\right)$ for all possible p_i where $i = 1, 2, 3, 4$.

It can be solved by using (30) and (37). Results are shown in Fig. 6. Here we have selected same $\hat{V}(x)$ as in the matched uncertainties case (Sect. 5.1(i)). The number of mesh points has been selected to run an offline algorithm presented in Sect. 4.1 is 2,500. NN-based robust optimal solution gives

$$\mathbf{W} = [0.4715;\ 0.5700;\ 0.6728;\ 0; 0.0001; 0.00015; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0]$$

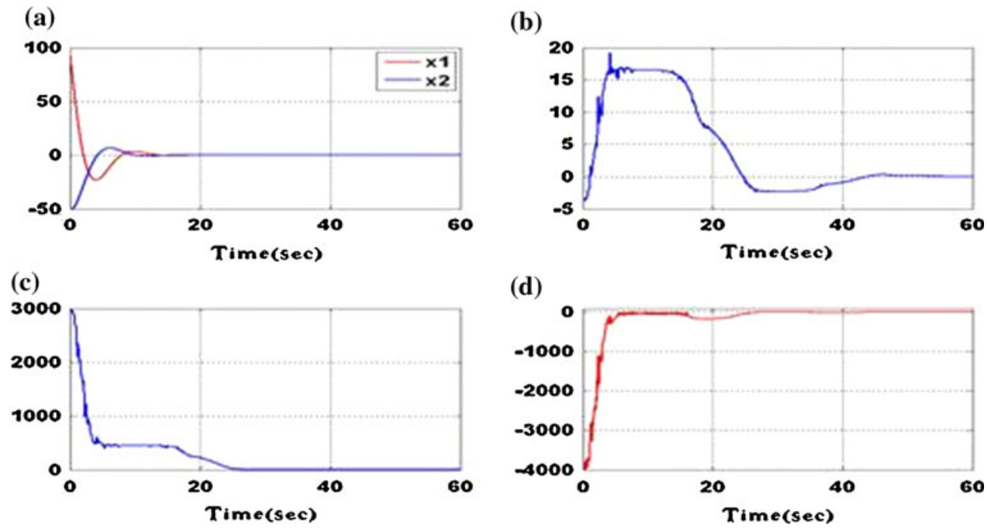

Fig. 6 (for $p = [-0.2\ -100\ 0\ -100]$) **a** System states versus time, **b** variation of control input, **c** Lyapunov function and **d** derivative of Lyapunov function

The robust control law can be found using (31). It remains bounded, i.e., $|u| \leq 50$. Simulation is carried out using time-varying parameters $p_1(t) = -0.095$, $p_2(t) = 100 \cdot \sin(t)$, $p_3(t) = 0.05$ and $p_4(t) = -50 \cdot \sin(2t)$. Auxiliary control found from the nominal model is $v = -0.12658x_1 - 0.62842x_2$. It is observed that v satisfies the sufficient condition, i.e., $2\rho^2\|v\|^2 \leq \beta^{*2}\|x\|^2$. Hence, it is proved that $\hat{u}$ is a robust control.

Figure 6a shows variation in all the system states. It has been observed that all the states converge to the equilibrium point. Variations in control signal are shown in Fig. 6b. It can be observed from the Fig. 6c, d that $\hat{V}(x) \geq 0$ and $\dot{\hat{V}}(x) \leq 0$ for all the values of x, which ensures $\hat{V}(x)$ is the Lyapunov function. It shows approximate value function, which is the solution of HJB equation. Verification of the conditions (22) and (23) of Theorem 2 can be observed from the Fig. 7. The boundedness of control input and convergence of the system state to the equilibrium point validate proposed algorithm.

6 Conclusions

The contribution of this paper is a methodology for designing bounded controllers for nonlinear uncertain systems. It addresses the class of uncertainties like matched, unmatched type and uncertainties in the input matrix. The proposed frame work is based on the optimality-based robust control approach. Specifically, a robust nonlinear control problem is transformed into a constrained optimal

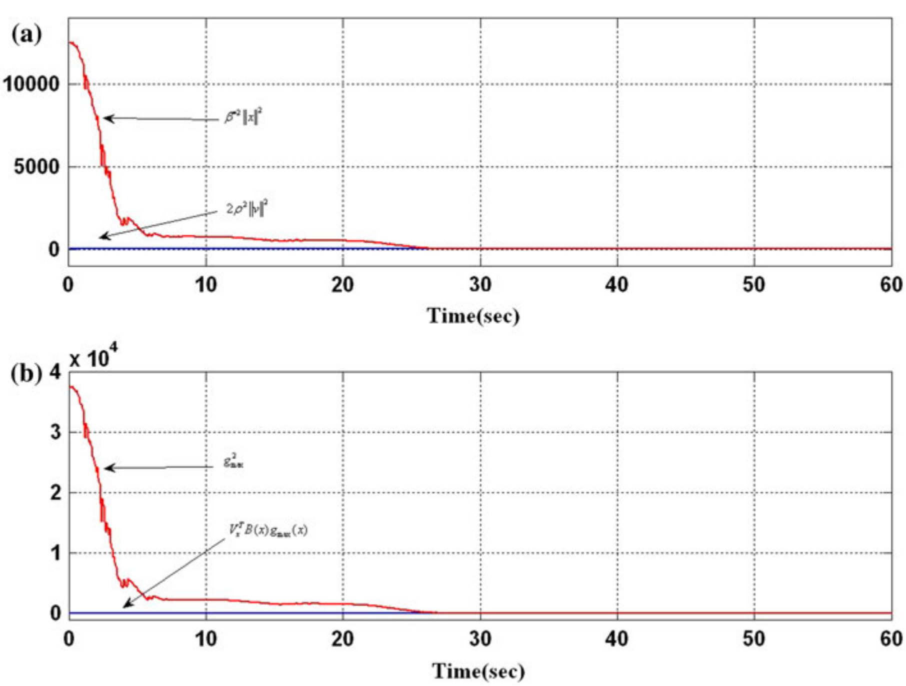

Fig. 7 **a** Verification of condition (22), **b** verification of condition (23)

control problem by modifying a cost functional to account for a class of uncertainties. The exact information about uncertainty is not required except some restrictive norm bound. We have adopted NN-based HJB solution to design robust-optimal control law that satisfies a prescribed bound. Modifications are done on the earlier approaches to handle constraints on the input and system uncertainties. Least squares–based method is used to find the solution of NN-based HJB equation. Simulation results show the good agreement with that of theoretical observations. It is also observed that control signal generated by NN remains bounded. Furthermore, it is shown that the Lyapunov function guaranteeing stability is the solution of HJB equation for the nominal system. Though there are many direct techniques available for the robust control problem, in the present paper an alternate approach is suggested using NN-based HJB solution. We explored the proposed approach to three different nonlinear systems with different class of uncertainties, i.e., matched system uncertainties, uncertainties in the input matrix and unmatched type system uncertainties. Difficulty faced in this approach was the selection of basis function of neural network. The choice of the basis function is guided by engineering experience and intuition. The proposed approach may be extended for the output feedback controller design.

Appendix 1

The dynamics of robot manipulator is given by

$$M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q) = \tau$$

where q consists of joint variables, τ is the generalized forces, $M(q)$ the inertia matrix, $V(q,\dot{q})$ the centripetal vector, $G(q)$ the gravity vector and $F(\dot{q})$ the friction vector. For simplicity, we denote $N(q,\dot{q}) = V(q,\dot{q}) + F(\dot{q}) + G(q)$. There are uncertainties in $M(q)$ and $N(q,\dot{q})$ due to, say, unknown load on the manipulator and unmodeled frictions.

The robot manipulator has two joint variables: two angles q_1 and q_2. The inertia matrix is

$$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where

$$\begin{aligned} M_{11} &= J_1 + m_1 r_1^2 + J_2 + m_2(l_1^2 + r_2^2 + 2l_1 r_2 \cos q_2) \\ &\quad + J_L + m_L(l_1^2 + r_2^2 + 2l_1 r_2 \cos q_2) \\ M_{12} &= M_{21} = J_2 + m_2(r_2^2 + l_1 r_2 \cos q_2) + J_L + m_L l_2^2 \\ M_{22} &= J_2 + m_2 r_2^2 + J_L + m_L l_2^2. \end{aligned}$$

For simplicity, we denote $N(q,\dot{q}) = V(q,\dot{q}) + F(\dot{q}) + G(q)$. There are uncertainties in $M(q)$ and $N(q,\dot{q})$ due to, say, unknown load on the manipulator and unmodeled frictions.

The centripetal vector is

$$V(q,\dot{q}) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where

$$V_1 = (m_2 l_1 r_2 + m_L l_1 l_2)(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)\sin q_2 \quad \text{and}$$
$$V_2 = (m_2 l_1 r_2 + m_L l_1 l_2)\dot{q}_1^2 \sin q_2.$$

The friction vector

$$F(q,\dot{q}) = \begin{bmatrix} b_1\dot{q}_1 \\ b_2\dot{q}_2 \end{bmatrix}.$$

Without loss of generality, we assume that gravity vector $G(q) = 0$; otherwise, it is easy to calculate the additional torque to balance the gravity. The value of the parameters is as follows:

$m_1 = 13.86$ oz; $m_2 = 3.33$ oz; $J_1 = 62.39$ oz-in/rad/s^2;
$J_2 = 16.70$ oz-in/rad/s^2; $r_1 = 6.12$ in;
$r_2 = 3.22$ in; $l_1 = 8$ in; $l_2 = 6$ in; $b_1 = 0.2$ oz-in/rad/s;
$b_1 = 0.5$ oz - in/rad/s

The load unknown, so we assume

$m_L = 10\varepsilon$ oz; $J_L = 60\varepsilon^2$ oz-in/rad/s^2

where $\varepsilon \in [0, 1]$.

With these values, we can calculate $M(q)$ and $N(q,\dot{q})$ as follows:

$$M(q) = \begin{bmatrix} 846 + 60\varepsilon^2 + 1000\varepsilon + 172\cos q_2 + 960\varepsilon\cos q_2 & 51 + 60\varepsilon^2 + 360\varepsilon + 86\cos q_2 \\ 51 + 60\varepsilon^2 + 360\varepsilon + 86\cos q_2 & 51 + 60\varepsilon^2 + 360\varepsilon \end{bmatrix}$$

$$N(q,\dot{q}) = \begin{bmatrix} (86 + 480\varepsilon)(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)\sin q_2 + 0.2\dot{q}_1 \\ (86 + 480\varepsilon)\dot{q}_1^2 \sin q_2 + 0.5\dot{q}_2 \end{bmatrix}$$

Assuming $\varepsilon = 1$ for the simplicity, we can select the following $M_0(q)$ and $N_0(q,\dot{q})$:

$$M_0(q) = \begin{bmatrix} 1906 + 1132\cos q_2 & 471 + 86\cos q_2 \\ 471 + 86\cos q_2 & 471 \end{bmatrix}$$

$$N_0(q,\dot{q}) = \begin{bmatrix} 566(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)\sin q_2 + 0.2\dot{q}_1 \\ 566\dot{q}_1^2 \sin q_2 + 0.5\dot{q}_2 \end{bmatrix}.$$

Appendix 2

In Sects. 5.1 and 5.2, we applied proposed algorithm on two different class of nonlinear systems. Now we would like to show that proposed algorithm will also work for the linear system:Consider a linear uncertain system

$$\dot{x} = Ax + Bu + Bfx$$

It is assumed that f is bounded by a *known* term, $f_{\max}$, i.e., $\|f\| \le f_{\max}$.

For the given open-loop system, our aim is to find a feedback control law $u = Kx$ such that the closed-loop system is global asymptotically stable for all admissible uncertainties f. With the performance function $\int_0^\infty \left(x^T \mu f_{\max}^2 I_{n\times n} x + x^T Qx + u^T Ru\right)\mathrm{d}t$, where $\mu > 0$ is design parameter, this problem can be solved using following HJB equation:

$$\mathrm{HJB}(V^*(x)) = x^T \mu f_{\max}^2 I_{n\times n} x + x^T Qx + u^T Ru + V_x^{*T}(Ax + Bu) = 0$$

where $V^*(x) = x^T Px$. It gives optimal control law $u^* = Kx = -\frac{1}{2}R^{-1}B^T V_x^*$ and corresponding algebraic Reccati equation (ARE) is

$$A^T P + PA - P^T BR^{-1}B^T P + Q + \mu f_{\max}^2 I_{n\times n} = 0 \tag{38}$$

u^* becomes a robust control law if

$$\left(\mu f_{\max}^2 I_{n\times n} - PBf_{\max}\right) \ge 0. \tag{39}$$

It is similar to condition (11) proposed for nonlinear system.

We have considered following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[p \quad p]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $p \in [-10, 1]$ is the uncertain term. We would like to design a robust control law $u = Kx$ such that the closed-loop system is stable for all $p \in [-10, 1]$.

Since $f = p$, it gives $f_{\max}^2 = \|p\|^2 \le 100$.

The corresponding ARE (I) is solved with $\mu = 2$ and identity matrices Q and R. It gives solution of ARE as

$$P = \begin{bmatrix} 16.2127 & 15.2127 \\ 15.2127 & 15.2127 \end{bmatrix}$$

Condition (II) is always satisfied as

$$\left(\mu f_{\max}^2 I_{n\times n} - PBf_{\max}\right) = \begin{bmatrix} 47.873 & 47.873 \\ 47.873 & 47.873 \end{bmatrix} \ge 0.$$

Hence, we have shown that condition similar to (11) is always satisfied for the linear system.

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