Published in IET Control Theory and Applications
Received on 8th July 2008
Revised on 29th October 2008
doi: 10.1049/iet-cta.2008.0288

ISSN 1751-8644

Fixed final time optimal control approach for bounded robust controller design using Hamilton–Jacobi–Bellman solution

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Abstract: In this study, an optimal control algorithm based on Hamilton–Jacobi–Bellman (HJB) equation, for the bounded robust controller design for finite-time-horizon nonlinear systems, is proposed. The HJB equation formulated using a suitable nonquadratic term in the performance functional to take care of magnitude constraints on the control input. Utilising the direct method of Lyapunov stability, we have proved the optimality of the controller with respect to a cost functional, that includes penalty on the control effort and the maximum bound on system uncertainty. The bounded controller requires the knowledge of the upper bound of system uncertainty. In the proposed algorithm, neural network is used to approximate the time-varying solution of HJB equation using least squares method. Proposed algorithm has been applied on the nonlinear system with matched and unmatched system uncertainties. Necessary theoretical and simulation results are presented to validate proposed algorithm.

1 Introduction

The design of a constrained optimal control using the solution of Hamilton–Jacobi–Bellman (HJB) equation is a challenging problem. Attempts have been made to find constrained control law using non-quadratic performance functional [1–3]. Similar work has been reported with finite horizon system in [4]. Time-varying HJB solution has been explored for optimal control problem in [5]. All these HJB-based optimal controller designs need exact information about the system model and nominal model. However, all practical control systems have to be robust with respect to model uncertainty such as unknown or partially known time-varying process parameters, exogenous disturbances etc. So model uncertainty needs to be considered during the time of controller design process to avoid the deterioration of nominal closed-loop performance. In other words, we need to design robust feedback control law to tackle the system uncertainty.

Although there exist many methods [6–9] to address the robust control problems, it has been recognised that the Lyapunov concept and the Hamilton–Jacobi theory are the major analytic paradigms for designing and analysing uncertain systems [10, 11]. Even though above mentioned works provide systematic method for robust controller design, they do not in general lead to controllers that are optimal with respect to a meaningful cost. Lin [12, 13] proposed a method to find robust control using the solution of well known HJB equation which is difficult to solve in practice. In these papers, existence of the solution of HJB equation is assumed, without mentioning any procedure to find it. An algorithm is also developed to find robust control law without any constraint (or bound) on the control input. An alternative realistic approach is to obtain approximate time-varying solution of HJB equation using NN, for feedback controller design for finite time horizon non-linear uncertain system having constrained input. However, fixed-final-time-constrained robust controller design using NN-based HJB solution has not been explored in the literature.

In this paper, the robust control problem is formulated into an optimal control problem by properly choosing cost functional. The solution of optimal control problem becomes a solution to the robust control problem with bounded control input. The cost functional is modified to account for system uncertainties and constraints on the input. Hence it can be referred as a bounded robust–optimal control design approach. The main contribution is to realise this approach with necessary theoretical justifications and implementation of it through neural network (NN). The least squares method is used to find tuning law of NN, which is used to approximate the time-varying solution of HJB equation. In addition to this we also explored proposed algorithm on the non-linear uncertain systems (with matched and unmatched type system uncertainties) to ensure asymptotic stability of the closed-loop system. Moreover it is shown that the time-varying Lyapunov function guaranteeing stability is a solution to the HJB equation for the nominal system. Convergence proof of the present work is supplemented by necessary theoretical and simulation results.

The paper is organised as follows: in Section 2, robust–optimal control framework has been described for system with matched and unmatched uncertainties and bounded input. In Section 3, NN-based HJB solution is used to find constrained robust–optimal control law. Stability issues with this formulation are also discussed and the theoretical results are presented in the form of a lemma and theorems. Solution of NN-based HJB equation found by least squares method is given in Section 4. Numerical examples are given in Section 5 for the validity of the approach. Proposed work is concluded in Section 6.

2 Robust–optimal control framework

Consider a non-linear system

$$\dot{x} = \bar{A}(x) + B(x)u$$

where $x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}^m$ is the control input. Each component of u is bounded by a positive constant λ, i.e.

$$|u_i| \leq \lambda \in \mathbb{R};\ i = 1, 2, \ldots, m \tag{1}$$

Suppose that the function $\bar{A}(x)$ is known only up to an additive perturbation which is bounded by a known function, and this perturbation is in the range of $B(x)$, i.e. $\bar{A}(x)$ can be written as $\bar{A}(x) = A(x) + B(x)f(x)$ with unknown $f(x)$. The condition that unknown perturbation be in the range space of $B(x)$ is called the matching condition, and can be incorporated by expressing the system as

$$\dot{x} = A(x) + B(x)u + B(x)f(x) \tag{2}$$

The function $B(x)f(x)$ models matched uncertainty in the system dynamics. The nominal model $A(x)$ and $B(x)$ are known with $A(0) = 0$ and $f(0) = 0$. This assumption ensures that the origin is the equilibrium point of system (2). It is assumed that function $f(x)$ is bounded by a known function, $f_{\max}(x)$:

$$|f(x)| \leq f_{\max}(x) \tag{3}$$

It is also assumed that the nominal system dynamics $A(x) + B(x)u$ is Lipschitz continuous on a set $\Omega \subseteq \mathbb{R}^n$ containing the origin and that the nominal system is stabilisable in the sense that there exists a continuous control on Ω that asymptotically stabilises the nominal system. In this paper we seek a constrained optimal control that will compensate for the perturbation, both for matched and unmatched system uncertainties.

2.1 System with matched uncertainty

In this section, design of an optimal control law is proposed to ensure global asymptotic stability of the system (2).

2.1.1 Robust control problem: For the open-loop system (2), find a feedback control law $u = K(x)$ such that the closed-loop system is globally asymptotically stable for all admissible uncertainties $f(x)$.

This problem can be formulated into an optimal control of the nominal system with appropriate cost functional.

2.1.2 Optimal control problem: For the nominal system

$$\dot{x} = A(x) + B(x)u \tag{4}$$

find a feedback control $u = K(x)$ that minimises the cost functional

$$\int_{t_0}^{t_f} \left(f_{\max}^2(x) + x^{\mathrm{T}}Qx + M(u)\right) \mathrm{d}t$$

where

$$M(u) = 2\int_0^u \lambda\ \tanh^{-1}(v/\lambda)R\,\mathrm{d}v = 2\lambda uR\ \tanh^{-1}(u/\lambda) + \lambda^2 R\ \ln(1 - u^2/\lambda^2) > 0 \tag{5}$$

is non-quadratic term expressing cost related to constrained input. The matrices Q and R are positive definite matrices showing the weightage of system states and control inputs, respectively.

In this paper, we address the following problems:

1. Solution of the robust control problem (2.1.1) and optimal control problem (2.1.2) are equivalent.

2. Solve the optimal control problem using NN. More specifically, HJB equation is solved using NN.

doi: 10.1049/iet-cta.2008.0288

To solve the optimal control problem, let

$$V(x_0, t_0) = \phi(x(t_f), t_f) + \min_u \int_{t_0}^{t_f} \left(f_{max}^2(x) + x^T Qx + M(u)\right) dt \quad (6)$$

be the minimum cost of bringing system (4) from initial condition x_0 to equilibrium point 0. The HJB equation gives

$$\min_u (f_{max}^2(x) + x^T Qx + M(u) + V_t + V_x^T (A(x) + B(x)u)) = 0 \quad (7)$$

where

$$V_x = \frac{\partial V(x, t)}{\partial x} \text{ and } V_t = \frac{\partial V(x, t)}{\partial t}$$

This is a time-varying partial differential equation (PDE) with $V(x, t)$ being the cost function for any given u and it is solved backward in time from $t = t_f$. By setting $t_0 = t_f$ in (6), its boundary condition is seen to be $V(x(t_f), t_f) = \phi(x(t_f), t_f)$. If $u = K(x)$ is the solution to the optimal control problem then according to Bellman's optimality principle [14], it can be found by solving the following HJB equation

$$\text{HJB}(V(x, t)) = f_{max}^2(x) + x^T Qx + M(u) + V_t + V_x^T (A(x) + B(x)u) = 0 \quad (8)$$

The optimal control law is computed by solving $\partial\text{HJB}(V(x, t))/\partial u = 0$

$$\Rightarrow \frac{\partial M(u)}{\partial u} + V_x^T B(x) = 0$$

Using (5) the above equation can be written as

$$2\lambda R \tanh^{-1}(u/\lambda) + V_x^T B(x) = 0 \quad (9)$$

$$\Rightarrow 2\lambda R \tanh^{-1}(-u/\lambda) = V_x^T B(x)$$

$$\Rightarrow u = K(x) = -\lambda \tanh\left(\frac{R^{-1}}{2\lambda} V_x^T B(x)\right) \quad (10)$$

Definition 1 (admissible controls): A control u, defined to be admissible with respect to (6) on Ω, denoted by $u \in \Gamma(\Omega)$ with u continuous on Ω, $u(0) = 0$, stabilises (4) on Ω, and $\forall x_0 \in \Omega$, $V(x(t_0), t_0)$ is finite.

With this basic introduction, following result is stated to show the equivalence of the solution of robust and optimal control problems.

Theorem 1: Consider the nominal system (4) with the performance function (5). Assume that there exists a function $V(x, t)$, the solution of HJB equation (8). Using this solution, bounded control law (10) ensures global asymptotic closed-loop stability of uncertain non-linear system (2) if the following condition is satisfied

$$f_{max}(x) \ge V_x^T B(x) \quad (11)$$

Proof: Here $u = K(x)$ is an optimal control law defined by (10) and $V(x, t)$ is the optimum solution of HJB equation (8). We now show that $u = K(x)$ is a solution to the robust control problem, i.e. the equilibrium point $x = 0$ of system (2) is globally asymptotically stable for all possible uncertainties $f(x)$. To do this it is shown that $V(x, t)$ is a Lyapunov function. Clearly, $V(x, t)$ is a positive definite function, i.e.

$$V(x, t) > 0, \ x \ne 0 \text{ and } t \ne 0 \text{ and } V(0) = 0$$

So

$$\begin{aligned}\dot{V}(x, t) &= (\partial V/\partial x)^T (dx/dt) + \partial V/\partial t \\ &= V_x^T(x)\left(A(x) + B(x)K(x) + B(x)f(x)\right) + \partial V/\partial t \\ &= V_x^T(x)(A(x) + B(x)K(x)) + \partial V/\partial t + V_x^T(x)B(x)f(x)\end{aligned}$$

Using (8), one can write

$$\begin{aligned}&= -f_{max}^2(x) - x^T Qx - M(u) + V_x^T(x)B(x)f(x) \\ &\le -x^T Qx - M(u) - \left(f_{max}^2(x) - V_x^T(x)B(x)f_{max}(x)\right)\end{aligned}$$

Using (11), one can write

$$\dot{V}(x, t) \le -x^T Qx \le 0$$

Thus conditions for Lyapunov local stability are satisfied. Consequently, there exists a neighbourhood $Z = \{x{:}\|x\| < p\}$ for some $p > 0$ such that if $x(t)$ enters Z, then $\lim_{t\to\infty} x(t) = 0$. But $x(t)$ cannot remain forever outside Z. Otherwise, $\|x\| \ge p$ for all $t \ge 0$.

Now define a scalar quantity $\alpha = \inf(x^T Qx) > 0$ such that $\|x\| \ge p$.

Therefore $V(x(t), t) - V(x(0), 0) = \int_0^t \dot{V}(x(\tau), \tau)\, d\tau \le -\int_0^t \alpha\, d\tau = -\alpha \int_0^t d\tau = -\alpha t$

So we have, $V(x(t), t) \le V(x(0), 0) - \alpha t \to -\infty$ as $t \to \infty$ which contradicts the fact that $V(x, t) > 0$ for $\forall x \ne 0$ and $t \ne 0$. Therefore $\lim_{t\to\infty} x(t) = 0$ no matter where the trajectory begins. □

Hence by knowing exact solution of HJB equation, one can find robust control law in the presence of matched uncertainties. In the next section, this approach is extended to the system having unmatched uncertainties.

doi: 10.1049/iet-cta.2008.0288

2.2 System having unmatched uncertainties

To generalise the robust–optimal control approach for handling the unmatched uncertainty, consider a non-linear system

$$\dot{x} = A(x) + B(x)u + C(x)f(x) \tag{12}$$

where $C(x)$ is a matrix of dimension $n \times q$ and $C(x) \neq B(x)$. It is assumed that $f(0) = 0$ and $A(0) = 0$; so that $x = 0$ is an equilibrium point of (14). It is also assumed that the nominal dynamics $A(x) + B(x)u$ is Lipschitz continuous on set $\Omega \subseteq \mathbb{R}^n$ containing the origin. It is required to find a bounded optimal control law, which ensures the closed-loop stability of the system (12) for unmatched uncertainties $f(x)$. Similar to Section 2.1, one can formulate this problem as

2.2.1 Robust control problem: For the open-loop system (12), find a feedback control law $u = K(x)$ such that the closed-loop system is globally asymptotically stable for all uncertainties $f(x)$ satisfying the following conditions:

i. $f(x)$ is bounded as defined in (3), i.e.

$$\|f(x)\| \leq f_{\max}(x)$$

ii. There exists a non-negative function $g_{\max}(x) \geq 0$ such that

$$\|B(x)^{+}C(x)f(x)\| \leq g_{\max}(x) \tag{13}$$

where + denotes the (Moore–Penrose) pseudo-inverse.

Decomposition of the uncertainty term $C(x)f(x)$ can be done as the sum of matched and an unmatched component by projecting $C(x)f(x)$ onto the range of $B(x)$. It can be written as

$$C(x)f(x) = B(x)B(x)^{+}C(x)f(x) + (I - B(x)B(x)^{+})C(x)f(x) \tag{14}$$

The above mentioned robust stabilisation problem can be formulated as the following optimal control problem.

2.2.2 Optimal control problem: Define an auxiliary system having unmatched uncertainty component of (14)

$$\dot{x} = A(x) + B(x)u + (I - B(x)B(x)^{+})C(x)v \tag{15}$$

where (u, v) is the control input.

Find a feedback control (u, v) that minimises the performance cost

$$\int_{t_0}^{t_f} \left(g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(u) + \rho^2\|v\|^2\right) dt$$

where ρ and β are some positive constants that serve as design parameters. Note that, the optimal control of system (15) has two components: $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^q$. It is very difficult to design a control law for system with unmatched uncertainties. To support control u, an augmented control v is added in the system to tackle the unmatched component of uncertainties defined by (14). However, in the actual system (12), only u component is used. An augmented control v plays an important role for proving asymptotic closed-loop stability of the system (14). It will be discussed in Theorem 2.

To solve optimal control problem, let

$$V(x_0, t_0) = \phi(x(t_f), t_f) + \min_{u,v}\int_0^{t_f}\left(g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(u) + \rho^2\|v\|^2\right) dt \tag{16}$$

to be the minimum cost of bringing the system (15) from initial condition x_0 to equilibrium point 0. The HJB equation gives us

$$\min_{u,v}\left(g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(u) + \rho^2\|v\|^2 + V_t + V_x^{\mathrm{T}}(A(x) + B(x)u + (I - B(x)B^{+}(x))C(x)v)\right) = 0$$

where

$$V_x = \frac{\partial V(x, t)}{\partial x} \text{ and } V_t = \frac{\partial V(x, t)}{\partial t}$$

This time-varying PDE is solved backward in time from $t = t_f$ with boundary condition $V(x(t_f), t_f) = \phi(x(t_f), t_f)$. If $u = K(x)$ is the solution to the optimal control problem, then according to Bellman's optimality principle [14], it can be found by solving the following HJB equation

$$\begin{aligned}\mathrm{HJB}(V(x,t)) = {} & g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(u) + \rho^2\|v\|^2 \\ & + V_t + V_x^{\mathrm{T}}(A(x) + B(x)u \\ & + (I - B(x)B^{+}(x))C(x)v) = 0\end{aligned} \tag{17}$$

which gives the optimal control

$$u = K(x) = -\lambda \tanh\left(\frac{R^{-1}}{2\lambda} V_x^{\mathrm{T}} B(x)\right) \tag{18}$$

and

$$v = -\frac{1}{2\rho^2} V_x^{\mathrm{T}}(I - \mathrm{BB}^{+})C \tag{19}$$

Not that, both controls admissible with respect to cost (16).

Following theorem proves that the control law (18) is the solution of the robust control problem.

IET Control Theory Appl., 2009, Vol. 3, Iss. 9, pp. 1183–1195
doi: 10.1049/iet-cta.2008.0288

Theorem 2: Consider the nominal system (15) with performance function (16). Assume that there exists a function $V(x, t)$, the solution of HJB (17), by properly choosing ρ and β. Using this solution, control law (18) ensures global asymptotic closed-loop stability of uncertain non-linear system (12) for some β^* such that $|\beta^*| < |\beta|$, if following conditions are satisfied

$$2\rho^2\|v\|^2 \le \beta^{*2}\|x\|^2, \ \forall x \in \mathbb{R}^n \tag{20}$$

$$g_{\max} \ge V_x^{\mathrm{T}}B, \ \forall x \in \mathbb{R}^n \tag{21}$$

Proof: It is proved here that using control law defined in (18) and (19), the system (12) remains globally asymptotically stable for all possible $f(x)$. To do this, we show that $V(x, t)$ is a Lyapunov function.

Clearly, $V(0) = 0$ and $V(x, t) > 0$ for $\forall x \neq 0$ and $t \neq 0$.

The time derivative of $V(x, t)$ is shown to be negative definite

$$\begin{aligned}\dot{V}(x, t) &= (\partial V/\partial x)^{\mathrm{T}}(\mathrm{d}x/\mathrm{d}t) + \partial V/\partial t \\ &= V_x^{\mathrm{T}}(A(x) + B(x)u + C(x)f(x)) + \partial V/\partial t \\ &= V_x^{\mathrm{T}}(A(x) + B(x)u + (I - B(x)B^+(x))C(x)v) \\ &\quad + B(x)B^+(x)C(x)f(x) + (I - B(x)B^+(x)) \\ &\qquad C(x)(f(x) - v)) + \partial V/\partial t \\ &= V_x^{\mathrm{T}}(A(x) + B(x)u + (I - B(x)B^+(x))C(x)v)) \\ &\quad + \partial V/\partial t + V_x^{\mathrm{T}}B(x)B^+(x)C(x)f(x) \\ &\quad + V_x^{\mathrm{T}}(I - B(x)B^+(x))C(x)(f(x) - v)\end{aligned}$$

Using (17) and (19), we have

$$\begin{aligned}\dot{V}(x, t) = &- g_{\max}^2(x) - \rho^2 f_{\max}^2(x) - \beta^2\|x\|^2 - M(u) - \rho^2\|v\|^2 \\ &+ V_x^{\mathrm{T}}B(x)B^+(x)C(x)f(x) - 2\rho^2 v^{\mathrm{T}}(f(x) - v)\end{aligned} \tag{22}$$

Since $-2\rho^2 v^{\mathrm{T}} f(x) \le \rho^2\left(\|v\|^2 + \|f(x)\|^2\right)$

Using (13) one can write

$$\begin{aligned}\dot{V}(x, t) &\le -M(u) - \left(g_{\max}^2 - V_x^{\mathrm{T}}B(x)g_{\max}(x)\right) \\ &\quad - \rho^2\left(f_{\max}^2 - \|f(x)\|^2\right) + 2\rho^2\|v\|^2 - \beta^2\|x\|^2 \\ &\le -M(u) - \left(g_{\max}^2 - V_x^{\mathrm{T}}B(x)g_{\max}(x)\right) \\ &\quad - \rho^2\left(f_{\max}^2 - \|f(x)\|^2\right) + 2\rho^2\|v\|^2 \\ &\quad - \beta^{*2}\|x\|^2 - (\beta^2 - \beta^{*2})\|x\|^2\end{aligned}$$

Using conditions (20) and (21)

$$\dot{V}(x, t) \le -(\beta^2 - \beta^{*2})\|x\|^2 < 0$$

Thus, the conditions of the Lyapunov stability theorem are satisfied. Using this result one can prove global stability similar to theorem 1. □

Theorems 1 and 2 are valid if we know the exact solution of HJB equation, which is a difficult problem. In the next section, NN is used to approximate value-function V which is the solution of HJB equation.

3 NN-based robust-optimal control

In this section we use NN to find approximate solution of HJB equation, which is utilised to find robust–optimal control. It is well known that an NN can be used to approximate smooth time-invariant functions on prescribed compact sets [15, 16]. It can be used to approximate a non-linear mapping. Let $\mathbb{R}$ denote the real numbers. Given $x_k \in \mathbb{R}$, define $x = [x_0, x_1, \ldots, x_n]^{\mathrm{T}}$, $y = [y_0, y_1, \ldots, y_m]^{\mathrm{T}}$ and weight matrices $W_L = [w_1, w_2, \ldots, w_L]^{\mathrm{T}}$. Then the ideal NN output can be expressed as $y = W_L^{\mathrm{T}}\sigma_L(x)$ with the vector of NN activation function $\sigma_L(x) = [\sigma_1(x), \sigma_2(x), \ldots, \sigma_L(x)]^{\mathrm{T}}$. It is assumed to be orthonormal and satisfy the NN approximation property [3]. In [17], it is shown that NN with time-varying weights can be used to approximate uniformly continuous time-varying functions. We assume that $V(x, t)$ is smooth and so uniformly continuous on a compact set.

Let the NN structure to approximate the value function $V(x, t)$ for $t \in [t_0, t_f]$ be defined as

$$\hat{V}(x, t) = \sum_{j=1}^{L} w_j(t)\sigma_j(x) = W_L^T(t)\sigma_L(x) \tag{23}$$

which gives

$$\hat{V}_x(x, t) = \frac{\partial \hat{V}}{\partial x} = W_L^{\mathrm{T}}(t)\frac{\partial \sigma_L(x)}{\partial x} = W_L^{\mathrm{T}}(t)\nabla\sigma_L(x) \tag{24}$$

and

$$\hat{V}_t(x, t) = \frac{\partial \hat{V}}{\partial t} = \sigma_L^{\mathrm{T}}(x)\frac{\partial W_L(t)}{\partial t} \tag{25}$$

where $W_L(t) = [w_1(t), w_2(t), \ldots, w_L(t)]^{\mathrm{T}}$ is the set of NN weights. $\sigma(x)$ is selected such that $\hat{V}(0) = 0$ and $\hat{V}(x, t) > 0$ for $\forall x \neq 0$ and $t \neq 0$. It is assumed that L is large enough so that $\hat{V}(x(t_f), t_f) = W_L^{\mathrm{T}}(t_f)\theta_L(t_f) = \sigma_L(x(t_f), t_f)$, i.e. there exist weights $W_L(t_f)$ that exactly satisfy the approximation at $t = t_f$. The set $\sigma_L(x)$ is selected to be independent. Then, without loss of generality, they can be assumed to be orthonormal, i.e. select equivalent basis functions to $\sigma_i(x)$ that are also orthonormal [18]. The orthonormality of the set $\{\sigma_i(x)\}_1^\infty$ on Ω implies that, for a real-valued function $\eta(x, t) \in \mathbb{R}$

$$\eta(x, t) = \sum_{j=1}^{\infty} \langle \eta(x, t), \sigma_j(x) \rangle_\Omega \sigma_j(x)$$

where $\langle f, g \rangle_\Omega = \int_\Omega g f^{\mathrm{T}} \, \mathrm{d}x$ is an outer product, f and g are continuous functions, and the series converges pointwise [19], i.e. for any $\mu > 0$ and $x \in \Omega$, one can choose N sufficiently large to guarantee that $\left| \sum_{j=N+1}^{\infty} \langle \eta(x, t), \sigma_j(x) \rangle_\Omega \sigma_j(x) \right| < \mu$ for all time $t \in [t_0, t_f]$.

Note that, since one requires $\partial V(x, t)/\partial t$ in (8) and (17), the NN weights are selected to be time varying [5]. This is similar to the method of selecting $\sigma(x)$ used for optimal control in [4]. For infinite time case, the NN weights are constant. The NN weights will be selected to minimise a residual error in least squares sense over a set of points sampled from a compact set Ω_0 inside the region of stability (RAS) of the initial stabilising control [4].This method will be discussed in Section 4 for solving robust control problem. With this background, we propose the robust–optimal control framework based on NN in the next section.

3.1 System with matched uncertainties

For the matched uncertainties case, the HJB equation with (23)–(25) can be written as

$$\mathrm{HJB}(\hat{V}(x, t)) = f_{\max}^2(x) + x^{\mathrm{T}} Q x + M(\hat{u}) + \hat{V}_{\mathrm{t}} + \hat{V}_x^{\mathrm{T}}(A(x) + B(x)\hat{u}) = e \qquad (26)$$

e represents an approximation error. If e is negligible, then (26) becomes similar to HJB (8) i.e.

$$\mathrm{HJB}(\hat{V}(x), t) = f_{\max}^2(x) + x^{\mathrm{T}} Q x + M(\hat{u}) + \hat{V}_{\mathrm{t}} + \hat{V}_x^{\mathrm{T}}(A(x) + B(x)\hat{u}) \approx 0 \qquad (27)$$

The optimal control law can be found by taking derivative of (27) w.r.t. $\hat{u}$. It can be found as

$$\hat{u}(x) = -\lambda \tanh\left(\frac{R^{-1}}{2\lambda} B(x)^{\mathrm{T}} \hat{V}_x(t) \right) = -\lambda \tanh\left(\frac{R^{-1}}{2\lambda} B(x)^{\mathrm{T}} W(t) \nabla \sigma^{\mathrm{T}}(x) \right) \qquad (28)$$

We introduce a lemma to show the existence of NN-based HJB solution for optimal control using modified performance functional. Theorem 3 shows the relationship between the robust control and the optimal control for NN-based HJB solution.

Lemma 1: Given $u \in \Gamma(\Omega)$, let $\hat{V}(x, t) = \sum_{j=1}^{L} w_j(t)\sigma_j(x)$ satisfy $\langle \mathrm{HJB}(\hat{V}(x, t)), \sigma(x) \rangle_\Omega = 0$ and $\langle \hat{V}(t_f), \sigma(x) \rangle_\Omega = 0$ on a compact set $\Omega \subset \mathbb{R}^n$, and let $V(x, t) = \sum_{j=1}^{\infty} c_j(t)\sigma_j(x)$ and $C = [c_1(t), c_2(t), \ldots, c_L(t)]^{\mathrm{T}}$ satisfy $\mathrm{HJB}(V(x, t)) = 0$ and $V(x(t_f), t_f) = \phi(x(t_f), t_f)$. If Ω is compact, $(f_{\max}^2(x) + x^{\mathrm{T}} x)$ is continuous on Ω and is in the space span $\{\sigma_j\}_1^\infty$, and if the coefficients $|w_j(t)|$ are uniformly bounded for all L, then $|\mathrm{HJB}(\hat{V}(x, t))| \to 0$ uniformly on Ω as L increases.

Proof: Cheng *et al.* [4] proposed a lemma for the existence of NN-based HJB solution for the optimal control problem. The existence of NN-based HJB solution for optimal control using modified performance functional can be proved on similar lines.

The hypothesis implies that $\mathrm{HJB}(\hat{V}(x), t)$ are in $L_2(\Omega)$. Note that

$$\langle \mathrm{HJB}(\hat{V}(x, t)), \sigma_j(x) \rangle_\Omega = \sum_{k=1}^{L} \dot{w}_k(t) \langle \sigma_k(x), \sigma_j(x) \rangle_\Omega + \sum_{k=1}^{L} w_k(t) \langle \nabla \sigma_k(x) A(x), \sigma_j(x) \rangle_\Omega + \sum_{k=1}^{L} w_k(t) \langle M(\hat{u}), \sigma_j(x) \rangle_\Omega + \langle \left(f_{\max}^2(x) + x^{\mathrm{T}} Q x \right), \sigma_j(x) \rangle_\Omega - \sum_{k=1}^{L} \langle w_k(t) \nabla \sigma_k(x) B(x) \lambda \times \tanh\left(\frac{1}{2\lambda} B^{\mathrm{T}}(x) \nabla \sigma_L^{\mathrm{T}}(x) W_L(t) \right), \sigma_j(x) \rangle_\Omega \qquad (29)$$

Since set $\{\sigma_j\}_1^\infty$ is orthogonal, $\langle \sigma_{\mathrm{k}}(x), \sigma_j(x) \rangle_\Omega = 0$.

Also, using (27) one can write

$$\left| \mathrm{HJB}(\hat{V}(x, t)) \right| = \left| \sum_{j=1}^{\infty} \langle \mathrm{HJB}(\hat{V}(x, t)), \sigma_j(x) \rangle_\Omega \sigma_j(x) \right| = \left| \sum_{j=L+1}^{\infty} \left(\sum_{k=1}^{L} w_k \langle \nabla \sigma_k(x) A(x), \sigma_j(x) \rangle_\Omega \right) \sigma_j(x) + \sum_{j=L+1}^{\infty} \left\langle \left(f_{\max}^2(x) + x^{\mathrm{T}} Q x \right), \sigma_j(x) \right\rangle_\Omega \sigma_j(x) + \sum_{j=L+1}^{\infty} \left(\sum_{k=1}^{L} w_k \langle M(\hat{u}), \sigma_j(x) \rangle_\Omega \right) \sigma_j(x) - \sum_{j=L+1}^{\infty} \left(\sum_{k=1}^{L} \langle w_k \nabla \sigma_k(x) B(x) \lambda \tanh \times \left(\frac{1}{2\lambda} B^{\mathrm{T}}(x) \nabla \sigma_L^{\mathrm{T}}(x) W_L \right), \sigma_j(x) \rangle \right) \sigma_j(x) \right|$$

doi: 10.1049/iet-cta.2008.0288

$$\leq \left| \sum_{k=1}^{L} w_k \sum_{j=L+1}^{\infty} \langle \nabla\sigma_k(x)A(x), \sigma_j(x)\rangle_\Omega \sigma_j(x) \right|$$
$$+ \left| \sum_{j=L+1}^{\infty} \left\langle \left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right)\sigma_j(x) \right\rangle_\Omega \sigma_j(x) \right|$$
$$+ \left| \left(\sum_{k=1}^{L} w_k \sum_{j=L+1}^{\infty} \langle M(\hat{u}), \sigma_j(x)\rangle_\Omega \sigma_j(x) \right) \right|$$
$$+ \left| \left(\sum_{k=1}^{L} w_k \sum_{j=L+1}^{\infty} \left\langle \nabla\sigma_k(x)B(x)\lambda \tanh \times \left(\frac{1}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L\right), \sigma_j(x) \right\rangle \sigma_j(x) \right) \right|$$

Hence

$$\left|\mathrm{HJB}(\hat{V}(x), t)\right| \leq \bar{P}\,\bar{Q}(x) + \bar{R}\,\bar{S}(x) + \bar{P}\,\bar{T}(x) + \left| \sum_{j=L+1}^{\infty} \left\langle \left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right)\sigma_j(x) \right\rangle_\Omega \sigma_j(x) \right|$$

where

$$\bar{P} = \max_{1 \leq k \leq L} |w_k|$$
$$\bar{Q}(x) = \sup_{x \in \Omega} \left| \sum_{k=1}^{L} \left(\sum_{j=L+1}^{\infty} \langle \nabla\sigma_k(x)\,A(x), \sigma_j(x)\rangle_\Omega \right) \sigma_j(x) \right|$$
$$\bar{R} = 1$$
$$\bar{S}(x) = \sup_{x \in \Omega} \left| \left(\sum_{k=1}^{L} \sum_{j=L+1}^{\infty} \langle M(\hat{u}), \sigma_j(x)\rangle_\Omega \sigma_j(x) \right) \right|$$
$$\bar{T}(x) = \sup_{x \in \Omega} \left| \left(\sum_{k=1}^{L} \sum_{j=L+1}^{\infty} \left\langle \nabla\sigma_k(x)B(x)\lambda \tanh \times \left(\frac{1}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L\right), \sigma_j(x) \right\rangle \sigma_j(x) \right) \right|$$

Suppose $\nabla\sigma_k(x)A(x)$, $M(\hat{u})$, $\nabla\sigma_k(x)B(x)\lambda \tanh\left(\frac{1}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L\right)$ and $\left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right)$ are continuous on Ω and are in $L_2(\Omega)$, and co-efficient $|w_j(t)|$ are uniformly bounded for all L, so the orthogonality of the set $\{\sigma_j\}_1^\infty$ implies that $\bar{Q}(x)$ and second and third terms on the right-hand side can be made arbitrarily small by an appropriate choice of L.

Therefore $\bar{P}\cdot\bar{Q}(x) + \bar{R}\cdot\bar{S}(x) + \bar{P}\cdot\bar{T}(x) \to 0$ and

$$\left| \sum_{j=L+1}^{\infty} \left\langle \left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right), \sigma_j(x) \right\rangle_\Omega \sigma_j(x) \right| \to 0$$

So, $\left|\mathrm{HJB}(\hat{V}(x), t)\right| \to 0$ uniformly on Ω as L increases. The details of justifying these arguments along with the necessary assumptions can be found in [2, 4, 20].

Lemma 1 shows the existence of NN-based HJB solution for the optimal control using cost functional $\int_0^{t_f} (f_{\max}^2(x) + x^{\mathrm{T}}Qx + M(\hat{u}))\,\mathrm{d}t$. Also approximate value function $\hat{V}(x)$ satisfies HJB equation (7) and it ensures $e \to 0$. □

Since Lemma 1 shows the existence of NN-based HJB solution, (26) can be written as (27). In the next theorem, equivalence of the NN-based solution of optimal control problem and robust control problem is proved.

Theorem 3: Assume that the NN-based HJB solution to the optimal control problem exists. Then control law defined by (28) ensures closed-loop asymptotic stability of non-linear uncertain system (2) if the following condition is satisfied:

$$f_{\max}(x) \geq \hat{V}_x^{\mathrm{T}} B(x) \tag{30}$$

Proof: Here $\hat{u}(x)$ is an optimal control law defined by (28) and $\hat{V}(x, t)$ is the solution of the HJB equation (27). We now show that with this control, the system remains asymptotically stable for all possible $f(x)$. Using definition (23) and from the selection of $\sigma(x)$, $\hat{V}(0, 0) = 0$ and $\hat{V}(x, t) > 0$ for $\forall x \neq 0$. Also $\dot{\hat{V}}(x, t) = \mathrm{d}\hat{V}/\mathrm{d}t < 0$ for $x \neq 0$ can be proved similarly as Theorem 1 by replacing $V(x, t)$ by $\hat{V}(x, t)$.

i.e.

$$\dot{\hat{V}}(x, t) = \left(\frac{\partial\hat{V}}{\partial x}\right)^{\mathrm{T}} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) + \frac{\partial\hat{V}}{\partial t}$$
$$= \hat{V}_x^{\mathrm{T}}(x)\big(A(x) + B(x)\hat{u} + B(x)f(x)\big) + \frac{\partial\hat{V}}{\partial t}$$
$$= \hat{V}_x^{\mathrm{T}}(x)(A(x) + B(x)\hat{u}) + \frac{\partial\hat{V}}{\partial t} + \hat{V}_x^{\mathrm{T}}(x)B(x)f(x)$$

Using (27), one can write

$$= -f_{\max}^2(x) - x^{\mathrm{T}}Qx - M(\hat{u}) + \hat{V}_x^{\mathrm{T}}(x)B(x)f(x)$$
$$\leq -x^{\mathrm{T}}Qx - M(\hat{u}) - \left(f_{\max}^2(x) - \hat{V}_x^{\mathrm{T}}(x)B(x)f_{\max}(x)\right)$$

Using (30), one can write

$$\dot{\hat{V}}(x, t) \leq -x^{\mathrm{T}}Qx \leq 0$$

Thus conditions for Lyapunov local stability are satisfied.

doi: 10.1049/iet-cta.2008.0288

Closed-loop global stability can be proved using similar argument mentioned in Theorem 1. □

From the above theorem, it can be proved that instead of solving the robust control problem, one can solve the optimal control problem. It is shown that this procedure always leads to a control which stabilises the system having matched uncertainty. In the next section, similar concept is explored for the system having unmatched uncertainties.

3.2 System having unmatched uncertainties

As described in Section 3.1 one can find robust–optimal control for the system with matched uncertainties using NN–HJB approach. In this section similar framework is extended for the system having unmatched uncertainties. Using the structure (23) of NN-based approximate value function the HJB equation can be written as

$$\mathrm{HJB}(\hat{V}(x,t)) = g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(\hat{u}) + \rho^2\|\hat{v}\|^2 + \hat{V}_t + \hat{V}_x^{\mathrm{T}}(A(x) + B(x)\hat{u} + (I - B(x)B^{+}(x))C(x)\hat{v}) = e \quad (31)$$

Here NN is used to approximate value function $V(x, t)$. Approximation error is represented by e. If e is negligible, then (31) becomes similar to (17). One can show the existence of NN-based HJB solution for the above performance functional by the similar kind of proof as in Lemma 1.

As $e \to 0$, we can write (31) as

$$\mathrm{HJB}(\hat{V}(x), t) = g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2 + M(\hat{u}) + \rho^2\|\hat{v}\|^2 + \hat{V}_t + \hat{V}_x^{\mathrm{T}}(A(x) + B(x)\hat{u} + (I - B(x)B^{+}(x))C(x)\hat{v}) \approx 0 \quad (32)$$

Approximate optimal control can be found by taking derivative of (32) w.r.t. $\hat{u}$ and $\hat{v}$

$$\hat{u}(x) = -\lambda \tanh\left(\frac{R^{-1}}{2\lambda} B(x)^{\mathrm{T}} \hat{V}_x(t)\right) = -\lambda \tanh \times \left(\frac{R^{-1}}{2\lambda} B(x)^{\mathrm{T}} W(t) \nabla\sigma^{\mathrm{T}}(x)\right) \quad (33)$$

and

$$\hat{v}(x) = -\frac{1}{2\rho^2}(I - \mathrm{BB}^{+})^{\mathrm{T}} C^{\mathrm{T}} W(t)\nabla\sigma^{\mathrm{T}}(x) \quad (34)$$

The relationship between robust control and optimal control for NN-based HJB solution can be defined similar to Theorems 2 and 3 for the unmatched uncertainty case. It can be observed for unmatched uncertainty case that, the robust control problem can be solved by solving corresponding optimal control problem. We have shown that this procedure always leads to a control law that stabilises the uncertain system. Next section is about the utilisation of the least squares method [21] for finding a HJB solution.

4 HJB solution by least-square method

Method of weighted residuals [4] was explored for optimal control problem. The unknown weights are determined by projecting the residual error e onto $\mathrm{d}e/\mathrm{d}\dot{W}$ and setting the result to zero using the inner product, i.e.

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}\dot{W}}, e \right\rangle = 0 \text{ for } \forall x \in \Omega \subseteq \mathbb{R}^n \quad (35)$$

where $\langle a, b\rangle = \int_\Omega ab\,\mathrm{d}x$ is a Lebesgue integral.

This method can be applied to solve robust–optimal control problem for the system having matched uncertainties. According to this method, by using definitions (24)–(26), we can write (35) as

$$\frac{\partial e(x, t)}{\partial \dot{W}(t)} = \sigma(x) \quad (36)$$

It can be written as

$$\langle -\dot{W}_L^{\mathrm{T}}(t)\sigma_L(x), \sigma_L(x)\rangle_\Omega + \langle -W_L^{\mathrm{T}}(t)\nabla\sigma_L(x)A(x), \sigma_L(x)\rangle_\Omega + \langle -\left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right), \sigma_L(x)\rangle_\Omega + \langle -M(\hat{u}), \sigma_L(x)\rangle_\Omega + \langle W_L^{\mathrm{T}}(t)\nabla\sigma_L(x)B(x)\lambda \tanh \times \left(\frac{1}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L(t)\right), \sigma_L(x)\rangle_\Omega = 0 \quad (37)$$

Hence weight updating law for the matched uncertainty case is

$$\dot{W}_L(t) = -\langle\sigma_L(x), \sigma_L(x)\rangle_\Omega^{-1}\langle\nabla\sigma_L(x)A(x), \sigma_L(x)\rangle_\Omega W_L^{\mathrm{T}}(t) - \langle\sigma_L(x), \sigma_L(x)\rangle_\Omega^{-1}\left\langle\left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right), \sigma_L(x)\right\rangle_\Omega - \langle\sigma_L(x), \sigma_L(x)\rangle_\Omega^{-1}\langle M(\hat{u}), \sigma_L(x)\rangle_\Omega + \langle\sigma_L(x), \sigma_L(x)\rangle_\Omega^{-1}\left\langle W^{\mathrm{T}}(t)\nabla\sigma_L(x)B(x)\lambda \tanh \times \left(\frac{R^{-1}}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L(t)\right), \sigma_L(x)\right\rangle_\Omega \quad (38)$$

The NN weights can be determined by integrating (38) backwards in time. Control law (29) can be found using these weights, which is the solution of robust control problem having matched uncertainties. Similar HJB solution can be found for unmatched uncertainties. Equation (34) can be

IET Control Theory Appl., 2009, Vol. 3, Iss. 9, pp. 1183–1195
doi: 10.1049/iet-cta.2008.0288

written using (24), (25) and (29) as

$$\langle -\dot{W}_L^{\mathrm{T}}(t)\sigma_L(x),\ \sigma_L(x)\rangle_\Omega + \langle -W_L^{\mathrm{T}}(t)\nabla\sigma_L(x)A(x),\ \sigma_L(x)\rangle_\Omega + \langle -(g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2),\ \sigma_L(x)\rangle_\Omega + \langle -(M(\hat{u}) + \rho^2\|\hat{v}\|^2),\ \sigma_L(x)\rangle_\Omega + \left\langle W_L^{\mathrm{T}}(t)\nabla\sigma_L(x)B(x)\lambda \tanh\left(\frac{1}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L(t)\right),\ \sigma_L(x)\right\rangle_\Omega = 0 \quad (39)$$

Hence weight updating law is

$$\dot{W}_L(t) = -\langle\sigma_L(x),\ \sigma_L(x)\rangle_\Omega^{-1}\langle\nabla\sigma_L(x)A(x),\ \sigma_L(x)\rangle_\Omega W_L^{\mathrm{T}}(t) - \langle\sigma_L(x),\ \sigma_L(x)\rangle_\Omega^{-1}\langle(g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2), \sigma_L(x)\rangle_\Omega - \langle\sigma_L(x),\ \sigma_L(x)\rangle_\Omega^{-1}\langle(M(\hat{u}) + \rho^2\|\hat{v}\|^2),\ \sigma_L(x)\rangle_\Omega + \langle\sigma_L(x),\ \sigma_L(x)\rangle_\Omega^{-1}\langle W_L^{\mathrm{T}}(t)\nabla\sigma_L(x)B(x)\lambda \tanh \times\left(\frac{R^{-1}}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L(t)\right),\ \sigma_L(x)\rangle_\Omega \quad (40)$$

By solving this equation, one can find control law using (33) which is the solution of robust control problem for the systems having unmatched uncertainties. In the next section proposed algorithm has been described.

4.1 Algorithm

Solving integration in (37) and (39) is computationally difficult, since evolution of the L_2inner product over Ω_0 is required. This can be addressed using collocation method [4, 20]. The integral can be well approximated by discretisation. A mesh of points of size Δx over integration region can be introduced on Ω_0. The terms of (38) and (40) can be rewritten as follows

$$A1 = \left[\sigma_L(x)|_{x_1}\ \ldots\ldots\ldots\ \sigma_L(x)|_{x_p}\right]^{\mathrm{T}};$$

$$A2 = \left[\nabla\sigma_L(x)A(x)|_{x_1}\ \ldots\ldots\ldots\ \nabla\sigma_L(x)A(x)|_{x_p}\right]^{\mathrm{T}};$$

$$A3 = \left[M(\hat{u})|_{x_1}\ \ldots\ldots\ldots\ M(\hat{u})|_{x_p}\right]^{\mathrm{T}}$$

(for matched uncertainties case)

$$A3 = \left[(M(\hat{u}) + \rho^2\|\hat{v}\|^2)|_{x_1}\ \ldots\ldots\ldots\ (M(\hat{u}) + \rho^2\|\hat{v}\|^2)|_{x_p}\right]^{\mathrm{T}}$$

for unmatched uncertainties case

$$A4 = \left[\nabla\sigma_L(x)B(x)\lambda \tanh\left(\frac{R^{-1}}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L(t)\right)|_{x_1}\ \nabla\sigma_L(x)B(x)\lambda \tanh\left(\frac{R^{-1}}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L(t)\right)|_{x_p}\right]^{\mathrm{T}}$$

$$A5 = \left[\left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right)|_{x_1}\ \ldots\ldots\ldots\ \left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right)|_{x_p}\right]^{\mathrm{T}}$$

(for matched uncertainties case);

$$A5 = \left[(g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2)|_{x_1}\ \ldots\ldots\ldots\ (g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2)|_{x_p}\right]^{\mathrm{T}}$$

for unmatched uncertainties case

where p represents number of points in the mesh. Reducing the mesh size, one can get the following results

$$\langle -\dot{W}_L^{\mathrm{T}}(t)\sigma_L(x), \sigma_L(x)\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -(A1^{\mathrm{T}}A1)\,\dot{W}_L(t)\,\Delta x;$$

$$\langle -W_L^{\mathrm{T}}(t)\nabla\sigma_L(x)A(x), \sigma_L(x)\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -(A1^{\mathrm{T}}A2)\,W_L(t)\,\Delta x;$$

$$\langle -M(\hat{u}), \sigma_L(x)\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -(A1^{\mathrm{T}}A3)\,\Delta x$$

(for matched uncertainties case)

$$\langle -(M(\hat{u}) + \rho^2\|\hat{v}\|^2), \sigma_L(x)\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -(A1^{\mathrm{T}}A3)\,\Delta x$$

(for unmatched uncertainties case)

$$\langle W_L^{\mathrm{T}}(t)\nabla\sigma_L(x)B(x)\lambda \tanh\left(\frac{1}{2\lambda}B^{\mathrm{T}}(x)\nabla\sigma_L^{\mathrm{T}}(x)W_L(t)\right), \sigma_L(x)\rangle_\Omega = \lim_{\|\Delta x\|\to 0}(A1^{\mathrm{T}}A4)W_L(t)\Delta x$$

$$\langle -\left(f_{\max}^2(x) + x^{\mathrm{T}}Qx\right), \sigma_L(x)\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -(A1^{\mathrm{T}}A5)\Delta x$$

(for matched uncertainties case)

$$\langle -(g_{\max}^2(x) + \rho^2 f_{\max}^2(x) + \beta^2\|x\|^2), \sigma_L(x)\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -(A1^{\mathrm{T}}A5)\Delta x$$

(for unmatched uncertainties case)

It implies that (37) and (39) can be written as

$$-(A1^{\mathrm{T}}A1)\dot{W}_L^{\mathrm{T}}(t) - (A1^{\mathrm{T}}A2)W_L^{\mathrm{T}}(t) - (A1^{\mathrm{T}}A3) + (A1^{\mathrm{T}}A4)W_L^{\mathrm{T}}(t) - (A1^{\mathrm{T}}A5) = 0 \quad (41)$$

It gives

$$\dot{W}_L(t) = -(A1^{\mathrm{T}}A1)^{-1}(A1^{\mathrm{T}}A2)W_L(t) - (A1^{\mathrm{T}}A1)^{-1}(A1^{\mathrm{T}}A3) + (A1^{\mathrm{T}}A1)^{-1}(A1^{\mathrm{T}}A4)W(t) - (A1^{\mathrm{T}}A1)^{-1}(A1^{\mathrm{T}}A5) \quad (42)$$

One can find the weights of NN by backward integrating (42) using final condition $W(t_{\mathrm{f}})$. It gives control law (29) (matched uncertainties case) and (33) (matched uncertainties case), which is the solution of the robust control problem for non-linear system with finite time horizon. In the next section, simulation experiments carried out on three systems to validate proposed algorithm are described.

5 Simulation experiments

5.1 Matched uncertainties case

In this section, we have explored proposed algorithm on a non-linear uncertain system.

Consider the non-linear system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1\left(x_1^2 + x_1^2\right)\\ \dot{x}_2 &= -x_1 + x_2 - x_1\left(x_1^2 + x_1^2\right) + u + px_1 \sin x_2\end{aligned} \quad (43)$$

where p is the unknown parameter and control input is bounded by $|u| \leq 1$. For simplicity let us assume that $p \in [-1, 1]$. It is in the matched uncertainty form, i.e.

$$\dot{x} = A(x) + B(x)u + B(x)f(x) \quad \text{with} \quad f(x) = px_1 \sin x_2.$$

Clearly, $|f(x)| \leq |x_1| = f_{\max}(x)$

Here our aim is to find the robust control law that will stabilise the system for all possible p.

This problem can be formulated into the following optimal control problem.

For the nominal system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1\left(x_1^2 + x_1^2\right);\\ \dot{x}_2 &= -x_1 + x_2 - x_1\left(x_1^2 + x_1^2\right) + u\end{aligned}$$

we have to find a feedback control law $u = K(x)$ that minimises

$$\int_0^{t_f} (f_{\max}^2(x) + x^T x + M(u))\,dt = \int_0^{t_f} (2x_1^2 + x_2^2 + 2\int_0^{\Phi} \tanh^{-1}(u)\,du)\,dt$$

where, $\Phi = -\tanh\left(1/2\ W_L^T(t)\nabla\sigma_L(x)B(x)\right)$. This problem can be solved by using the algorithm described in Section 4.1. Scalar parameter $p = 1$ has been selected for the purpose of simulating the plant. Here we have selected

$$\begin{aligned}\hat{V}(x, t) = {}& w_1(t)x_1^2 + w_2(t)x_2^2 + w_3(t)x_1x_2 + w_4(t)x_1^4 + w_5(t)x_2^4\\ &+ w_6(t)x_1^3x_2 + w_7(t)x_1^2x_2^2 + w_8(t)x_1x_2^3 + w_9(t)x_1^6\\ &+ w_{10}(t)x_2^6 + w_{11}(t)x_1^5x_2 + w_{12}(t)x_1^4x_2^2 + w_{13}(t)x_1^3x_2^3\\ &+ w_{14}(t)x_1^2x_2^4 + w_{15}(t)x_1x_2^5 + w_{16}(t)x_1^8 + w_{17}(t)x_2^8\\ &+ w_{18}(t)x_1^7x_2 + w_{19}(t)x_1^6x_2^2 + w_{20}(t)x_1^5x_2^3 + w_{21}(t)x_1^4x_2^4\\ &+ w_{22}(t)x_1^3x_2^5 + w_{23}(t)x_1^2x_2^6 + w_{24}(t)x_1x_2^7\end{aligned}$$

This is a NN with polynomial activation function and hence $\hat{V}(0) = 0$. It is a power series NN of 24 activation functions containing powers upto eighth order of the state variables of the system. Selecting the NN structure for approximating $V(x)$ is usually a natural choice guided by engineering experience and intuition. The number of neurons required is chosen to guarantee the uniform convergence of the algorithm. Neurons with eighth order power of the states variables were selected because for neurons with sixth power of the states, the algorithm did not converge. Higher-order power terms were producing similar results without much improvement. Hence to avoid computational complexity we have taken activation function up to eighth order. The activation functions for the NN selected in this paper satisfy the properties of the activation function discussed in [2]. NN-based HJB solution can be found using least squares method as described in the algorithm. All the weights are determined by backward integrating (42). For that purpose we have selected $t_f = 50$ and $W_L(t_f)$ is selected as follows:

$$\begin{aligned}W_L(t_f) = [&30.0463\ 10.7230\ 21.0872\ 33.8288\ 1.3228\ 41.3044\\ &28.2090\ 2.5712\ 19.8882\ 2.3689\ 40.8138\ 41.7514\\ &10.0255\ 4.7980\ 4.9658\ 3.5467\ 0.4724\ \ 13.5306\\ &20.6493\ 10.3097\ 0.0601\ 4.4958\ 3.5604\ 0.6157]\end{aligned}$$

Required quantities $A1$, $A2$, $A3$, $A4$ and $A5$ are evaluated for 1000 points in Ω_0. The states and control can be determined by forward integrating (43) and using these weights in (29).

Response of the robust controller is shown in Fig. 1. As shown in Fig. 1*a*, robust–optimal control approach shows convergence of system states to the equilibrium point. It can be observed from Fig. 1*c* that the time derivative of Lyapunov function remains negative for all the $t \in [0, t_f]$ and $x \in \Omega_0$. It shows valid approximation of the solution of HJB equation with NN. Variation in the control signal is shown in Fig. 1*b*. Here control signal generated by NN remains bounded, i.e. $|u| \leq 1$. It also shows that constrained robust–optimal control input converges to zero when the system is stabilised. In Fig. 1*d*, condition (30) is verified.

5.2 Unmatched uncertainties case

Let the non-linear system be defined by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u + \begin{bmatrix}1\\ 0\end{bmatrix}f(x_1, x_2)$$

doi: 10.1049/iet-cta.2008.0288

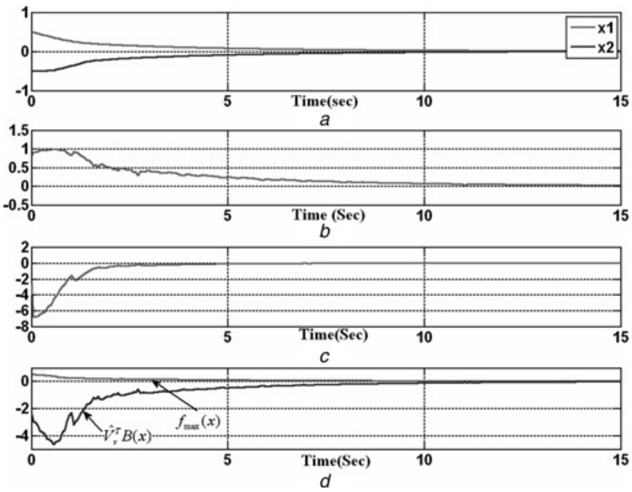

Figure 1 *Response of system with matched uncertainties*

a System states against time
b Variation of control input
c Derivative of Lyapunov function
d Varification of condition (30)

where

$$f(x_1, x_2) = 5p_1 x_1 \cos\left(\frac{1}{x_2 + p_2}\right) + 5p_3 x_2 \sin(p_4 x_1 x_2)$$

and $p_1(t) \in [-0.2, 0.01]$, $p_2(t) \in [-100, 100]$, $p_3(t) \in [-0.05, 0.05]$, $p_4(t) \in [-100, 0]$ are time-varying uncertainties and the control input is bounded by $|u| \le 50$. It is in the unmatched uncertainty form, i.e. $\dot{x} = A(x) + B(x)K(x) + C(x)f(x)$

Therefore $\|f(x_1, x_2)\|^2 \le x_1^2 + x_2^2 = f_{\max}^2(x)$ and $\|B^+ Cf(x_1, x_2)\|^2 = 0 = g_{\max}^2(x)$.

Also, $B^+ = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}} = B^{\mathrm{T}} = [01]$ and $(I - BB^+)C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$ $\rho = 1$ and $\beta = 1$ selected for the purpose of the simulation. As per description in Sections 2 and 3, the corresponding optimal control problem is as follows:

For the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} v$$

find a feedback control law that minimises the cost

$$\int_0^{t_f} \left(f_{\max}^2(x) + \rho^2 g_{\max}^2(x) + \beta^2\|x\|^2 + \rho^2\|v\|^2 + M(u)\right) \mathrm{d}t$$
$$= \int_0^{t_f} \left(2x_1^2 + 2x_2^2 + v^{\mathrm{T}}v + 100\int_0^{\Phi} \tanh^{-1}(u/50)\,\mathrm{d}u\right) \mathrm{d}t$$

where

$$\Phi = -50\tanh\left(\frac{1}{100} W_{\mathrm{L}}^{\mathrm{T}} \nabla\sigma_{\mathrm{L}}(x) B(x)\right)$$
$$\times \text{ for all possible } p_i \text{ where } i = 1, 2, 3, 4$$

It can be solved by using (42). Here we have selected the same $\hat{V}(x)$ as in the matched uncertainties case. For the simulation purpose we have selected, $W_L(t_f) = [4.364;\ 8.1335;\ 8.8854;\ 0;\ 0.0001;\ -0.0002;\ -0.0003;\ -0.0002;\ 0;\ 0;0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;]$; Required quantities $A1, A2, A3, A4$ and $A5$ are evaluated for 1000 points in Ω_0.

Response of the robust controller is shown in Fig. 2*a*. All the system states converge to equilibrium point. As shown in Fig. 2*b*, the control input remains bounded, i.e. $|u| \le 50$. Simulation is carried out using time varying parameters $p_1(t) = -0.095$, $p_2(t) = 100\ \sin(t)$, $p_3(t) = 0.05$ and $p_4(t) = -50\ \sin(2t)$. It can be observed from Fig. 2*c* that $\dot{\hat{V}}(x) \le 0$ for all $t \in [0, t_f]$ and $x \in \Omega_0$, which ensures $\hat{V}(x)$ is the Lyapunov function. It shows that approximated value function is the solution of HJB equation. In Fig. 2*d* verification of condition (21) is shown using approximate value function. The boundedness of control input and convergence of the system state to the equilibrium point validates proposed algorithm.

5.3 Optimal control of non-linear chained form system

In this section we show that an optimal control problem can be solved by the proposed algorithm. Consider a non-holonomic system converted to chained form [4] as

$$\dot{x}_1 = u_1;\ \dot{x}_2 = u_2 \text{ and } \dot{x}_3 = x_1 u_2$$

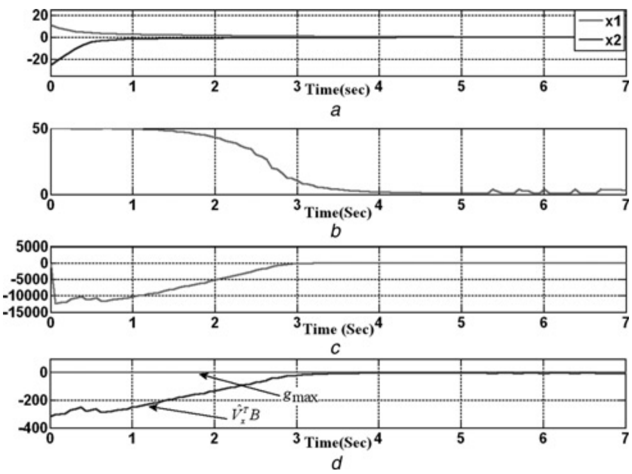

Figure 2 *Response of system with unmatched uncertainties*

a System states against time
b Variation of control input
c Derivative of Lyapunov function
d Verification of condition (21)

doi: 10.1049/iet-cta.2008.0288

It can be arranged in the matched uncertainty form, i.e.

$$\dot{x} = A(x) + B(x)u + B(x)f(x)$$

with $A(x) = [000; 000; 000]; B(x) = [10; 01; 0x_1]$ and $f(x) = [0; 0; 0]$ Clearly, $f_{max}(x) = 0$.

The optimal control problem can be solved by selecting Q and R as identity matrices of appropriate dimensions. Optimal control law can be found using (42). Here we have selected

$$\begin{aligned}\hat{V}(x, t) = {} & w_1(t)x_1^2 + w_2(t)x_2^2 + w_3(t)x_3^2 + w_4(t)x_1x_2 \\ & + w_5(t)x_2x_3 + w_6(t)x_1x_3 + w_7(t)x_1^4 + w_8(t)x_2^4 \\ & + w_9(t)x_3^4 + w_{10}(t)x_1^2x_2^2 + w_{11}(t)x_1^2x_3^2 \\ & + w_{12}(t)x_2^2x_3^2 + w_{13}(t)x_1^2x_2x_3 + w_{14}(t)x_1x_2^2x_3 \\ & + w_{15}(t)x_1x_2x_3^2 + w_{16}(t)x_1^3x_2 + w_{17}(t)x_1^3x_3 \\ & + w_{18}(t)x_1x_2^3 + w_{19}(t)x_1x_3^3 + w_{20}(t)x_2x_3^3 \\ & + w_{21}(t)x_2^3x_3\end{aligned}$$

This is a NN with polynomial activation function and hence $\hat{V}(0) = 0$. It is a power series NN of 21 activation functions containing powers upto fourth order of the state variables of the system. Neurons with fourth-order power of the state variables were selected because for neurons with second power of the states, the algorithm did not converge. Higher-order power terms were producing similar results without much improvement. Hence to avoid computational complexity we have taken activation function up to fourth order. In this example, $W_L(t_f) = [11; 4; 12.3784; 5.8192; 14.3095; 2.6897; 9.7010; 0.6699; 14.8043; 0; 12.7808; 0; 13.2514; 0.5419; 0; 13.2797; 0.7; 10; 5; 0; 7.4910]$ is selected for the simulation purpose. Required quantities $A1$, $A2$, $A3$, $A4$ and $A5$ are evaluated for 1000 points in Ω_0.

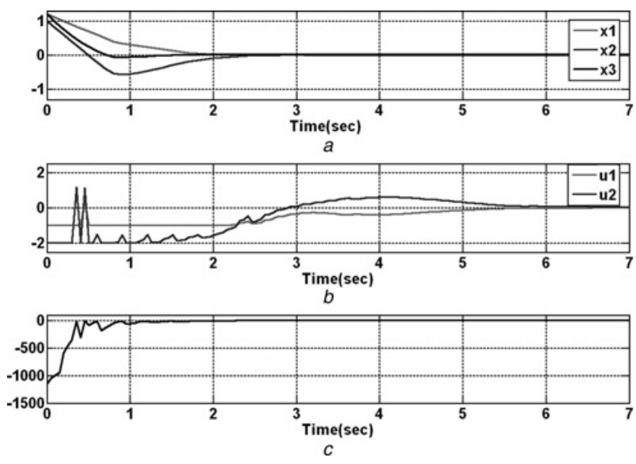

Figure 3 *Response of nonlinear chained-form system*

a System states against time
b Variation of control input
c Derivative of Lyapunov function

It can be observed from Fig. 3*a* that robust–optimal control approach shows convergence of system states to the equilibrium point. It can be observed from Fig. 3*c* that the time derivative of Lyapunov function remains negative for all $t \in [0, t_f]$ and $x \in \Omega_0$. It shows valid approximation of the solution of HJB equation with NN. Variation in the control signal is also shown in Fig. 3*b*. It remains bounded, i.e. $|u_1| \le 1$ and $|u_2| \le 2$. It also shows that constrained robust–optimal control input converges to zero when the system is stabilised.

6 Conclusions

The contribution of this paper is a methodology for designing bounded controllers for non-linear uncertain systems. It addresses a class of matched and unmatched uncertainties. The proposed framework is based on the optimality-based robust control approach. Specifically, a robust non-linear control problem is transformed into a constrained optimal control problem by modifying the cost functional to account for a class of uncertainties. The exact information about uncertainty is not required; some restrictive norm bound is only needed. We have adopted NN-based time-varying HJB solution to design robust–optimal control law that satisfies a prescribed bound on uncertainties, taken care of constraints on the input. Least squares-based method is used to find the solution of NN-based HJB equation. Simulation results on three different non-linear systems show a good agreement with that of theoretical observations. It is also observed that control signal generated by NN remains bounded. Furthermore, it is shown that the Lyapunov function guaranteeing stability is the time-varying solution of HJB equation for the nominal system. The selection of basis function of NN is guided by engineering experience and intuition. The proposed approach may be extended for the output feedback controller design.

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doi: 10.1049/iet-cta.2008.0288