Comparative Study for Various Fractional Order System Realization Methods

***Abstract*-- There are many dynamic systems that can be characterized better by using non-integer order dynamic model based on fractional calculus or, differentiation or integration. Traditional calculus is based on integer order differentiation and integration. In this paper, we have represented comparative study of different fractional order systems using different realization methods. Basic definitions of fractional calculus and fractional order dynamic systems are presented first. Additionally, different fractional order systems are introduced and commented. Numerical methods for simulating fractional order systems are given in detail. Also Discretization techniques for fractional order operators are explained in details. Both digital and analog realization methods of fractional order systems are introduced. Comparative study has been carried out for the time response analysis of fractional order system using various approximation methods. Finally, remarks on future research efforts in fractional order control are given.**

***Index Terms*–Fractional calculus, fractional order system, realization methods.**

I. INTRODUCTION

Fractional calculus was first come in existence about more than 300 years ago. In the year 1823 the fractional calculus was first applied by Abel. Now a day's fractional calculus has been widely used in many applications. By using these mathematical phenomena a real object can be described more accurately than the classical "integer-order" methods []] - [2]. Earlier peoples were using the integer-order models because of the absence of solution methods for fractional differential equations but now there are so many methods available for approximation of fractional derivative and integral. From some real world examples we can say that fractional order control can be used everywhere provided that the dynamic system has distributed parameters.

II. FRACTIONAL CALCULUS

The fractional calculus has been in existence since the development of integer-order calculus was done. It was first founded by Leibniz and L'Hˆopital probably in 1695 when the question of half-order or fractional order derivative was raised. Fractional calculus is a generalized form of integration and differentiation of non-integer order fundamental operator $_aD^r_t$, where a and t represents the limits of the operation. The basic definition of continuous integro-differential operator is given as follows:

$$_aD^r_t = \begin{cases} \frac{d^r}{dt^r} & , \Re(r) > 0 \\ 1 & , \Re(r) = 0 \\ \int_a^t (d\tau)^{-r} & , \Re(r) < 0 \end{cases}$$

Where r denotes the order of the operation, mostly r ∈ R but it can be a complex number also [3].

A. BASIC PROPERTIES OF FRACTIONAL CALCULUS

The main properties of fractional derivatives and fractional integrals are listed as below [4]-[5].

a) The fractional order derivative $_0D^\alpha_t$ f (t) is an analytical function of *z* and α If f (t) is an analytical function of t.
b) The result of operation $_0D^\alpha_t$ f(t) for noninteger order is same as result of differentiation of integer order n, when α = n, where n indicates an integer.
c) The operation $_0D^\alpha_t$ f (t) is the identity operator when α = 0. $_0D^0_t f(t) = f(t)$
d) Noninteger order differentiation and integration are two linear operations.
$$_0D^\alpha_t\, a\, f(t) + b\, g(t) = a\, _0D^\alpha_t f(t) + b\, _0D^\alpha_t g(t).$$
e) For the fractional-order integrals of arbitrary order, as mentioned earlier ℜ (α) > 0 and ℜ (β) > 0, it holds the additive law or in other words we can say semi group property. $_0D^\beta_t + {_0D^\alpha_t} = {_0D^{\alpha+\beta}_t}$

III. FRACTIONAL ORDER SYSTEMS AND STABILITY CRITERIA

A continuous fractional-order system can be described by the following fractional order differential equation [6].

$$a_n D^{\alpha n} y(t) + a_{n-1} D^{\alpha n-1} y(t) + \dots\dots\dots + a_0 D^{\alpha 0} y(t) = b_m D^{\beta m} u(t) + b_{m-1} D^{\alpha m-1} u(t) + \dots + bD^{\beta 0} u(t) \quad \dots. (1)$$

Where, $D^\alpha = {_0D^\alpha_t}$, $D^\beta = {_0D^\beta_t}$
an (n = 0, ⋯ N) and
bn (n = 0, ⋯ m) are constants;
αk (k =0, ⋯ n) and
βk (k =0, ⋯ m) are arbitrary real numbers.

Now to get the discrete model of the same fractional-order system as shown in equation (1), we have to use discrete approximations of the fractional-order operators so that we can obtain a general equation for the discrete transfer function of the proposed system [7].

$$G(z) = \frac{b_m\,(w(z^{-1}))^{\beta m} + \dots\dots + b_0\,(w(z^{-1}))^{\beta 0}}{a_n\,(w(z^{-1}))^{\alpha n} + \dots\dots + a_0\,(w(z^{-1}))^{\alpha 0}} \quad \dots.(2)$$

Where, $w(z^{-1})$ indicates the discrete equivalent of the Laplace transform of s.

A. Now the stability criteria for step and impulse responses for time domain are as follows[8]:

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- If $|\arg(\lambda_k)| \geq \alpha\pi$ then the response would be monotonically decreasing.
- If $\alpha\pi/2 < |\arg(\lambda_k)| < \alpha\pi$, then the response would be oscillatory with decreasing amplitude.
- If $|\arg(\lambda k)| = \alpha\pi/2$ then the response would be oscillatory with constant amplitude.
- If $|\arg(\lambda k)| < \alpha\pi/2$ or $|\arg(\lambda k)| \neq 0$ then the response would be oscillatory with increasing amplitude.
- If $|\arg(\lambda k)| = 0$ then the response would be monotonically increasing.

B. Now the basic equations and stability criteria for step and impulse responses for frequency domain are as follows[8]:

For the commensurate order systems the frequency response can be obtained by the addition of the individual terms of order α, which is the result of factorization of the specified function. Consider the equation given below.

$$G(s) = \frac{P(s^{\alpha})}{Q(s^{\alpha})} = \frac{\prod_{k=0}^{m} (s^{\alpha} + z_k)}{\prod_{k=0}^{m} (s^{\alpha} + \lambda_k)} \;, \; zk \neq \lambda k.$$

For each of these terms, referred to as $(s^{\alpha} + \gamma)^{\pm 1}$, the magnitude curve will have a slope which starts at zero and for higher frequencies it will tends to ±α20 dB/dec and the phase plot will tends to 0 to ±απ/2.

IV. DIFFERENT TYPES OF REALIZATION METHODS AND THEIR COMPARISION.

There are different approximation methods available for continuous and discrete time implementation of fractional order operator as given below [9]-[10].

A. Continuous time approximation methods are as follows:

1) Low-frequency continued fraction expansion
2) High-frequency continued fraction expansion
3) Carlson's method
4) Matsuda's method
5) Oustaloup recursive approximation
6) Modified oustaloup approximation

B. Discrete time approximation methods are as follows:

1) Zoh - zero order hold
2) Foh - linear interpolation of
3) Tustin - bilinear approximation
4) Prewarp - tustin approximation with frequency prewarping

Now we will see the continuous time implementation of different fractional order systems. All simulations are performed in using MATLAB software .

1. Using low frequency CFE, High frequency CFE, Carlson's and Matsuda's approximation method.

A. For transfer function $\frac{1}{s^{0.5}}$, corresponding frequency responses, using nid( ) function of N-integer toolbox in MATLAB is as follows[08]:

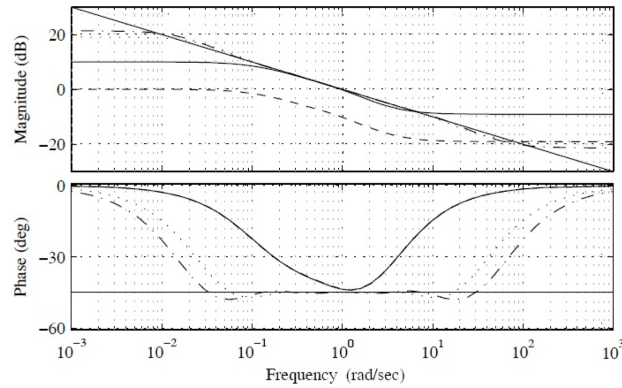

Figure 1.1 **C**omparisons of different methods [08]

2. Using oustaloup recursive method [11].

1) For transfer function $\frac{1}{s^{1.45}}$, corresponding time and frequency responses are as follows:

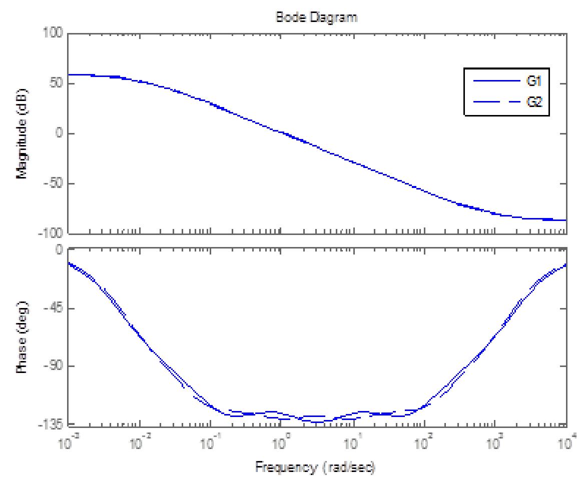

Figure 2.1.1 frequency responses of a fractional order integrator of order 1.45 with the Oustaloup approximation, solid lines for G1(s), dashed lines for G2(s).

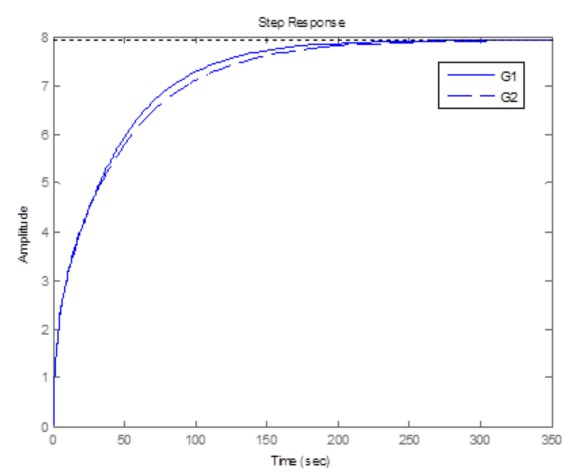

Figure 2.1.2 Time response of a fractional order integrator of order 1.45 with the Oustaloup approximation, solid lines for G1(s), and dashed lines for G2(s).

2. Using modified oustaloup method [12]

1) For transfer function $\frac{1}{s^{1.45}}$, corresponding time and frequency responses are as follows:

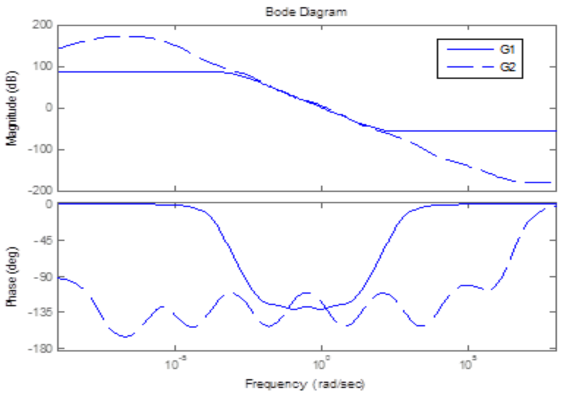

Figure 3.1.1 Frequency response of a fractional order integrator of order 1.45 using modified oustaloup approximations, solid lines for G1(s), and dashed lines for G2(s).

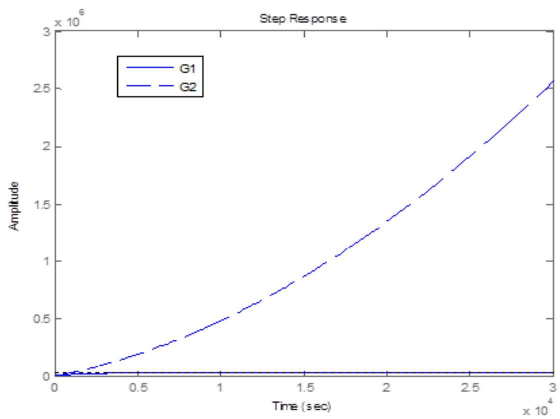

Figure 3.1.2 Time response of a fractional order integrator of order 1.45 using modified oustaloup approximations, solid lines for G1(s), and dashed lines for G2(s).

From the above simulation results we can observe that in the results of low frequency CFE, high frequency CFE, Carlson and Matsuda's methods, the fitting ranges are rather small and the quality of fit is not satisfactory. Whereas fitting quality of oustaloup method is highly superior and modified oustaloup is good for frequency range of interest. From this it is clear that oustaloup gives better performance than other approximations.

Now we will see the discrete time implementation of different fractional order systems using oustaloup and modified oustaloup method.

A. Using zoh, foh, methods and oustaloup approximation for transfer function $\frac{1}{s^{1.45}}$, corresponding time and frequency responses are as follows [13]-[14]:

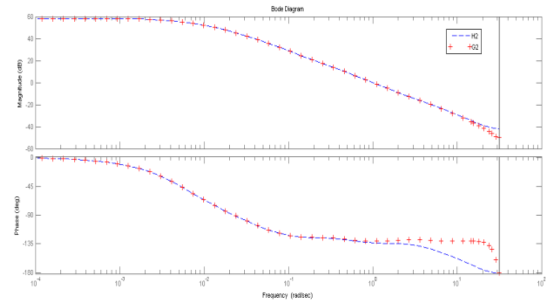

Figure 2.1 **Frequency** response of a fractional order integrator of order 1.45 using oustaloup approximations with zoh (H2) and foh (G2) methods.

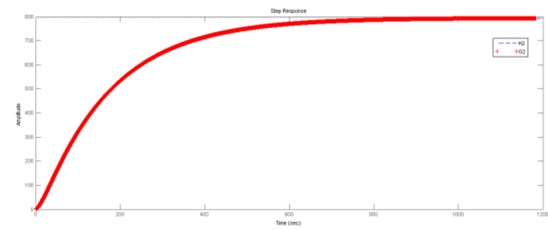

Figure 2.2 **Time** response of a fractional order integrator of order 1.45 using oustaloup approximations with zoh (H2) and foh (G2) methods.

Using Tustin, prewarp and matched methods and oustaloup approximation for transfer function $\frac{1}{s^{1.45}}$, corresponding time and frequency responses are as follows:

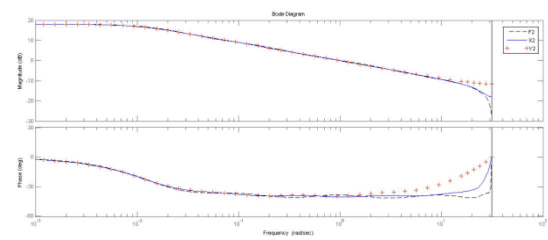

Figure 2.3 **Frequency** response of a fractional order integrator of order 1.45 using oustaloup approximations with Tustin (F2), prewarp(X2) and matched (Y2) methods.

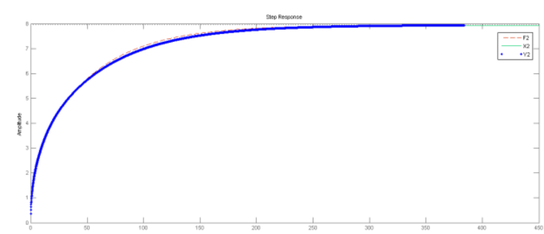

Figure 2.4 **Time** response of a fractional order integrator of order 1.45 using oustaloup approximations with Tustin (F2), prewarp(X2) and matched (Y2) methods.

Rise time for tustine method is 110s and rise time of prewarp and matched method is 120s.

B. Using zoh, foh method and modified oustaloup approximation for transfer function $\frac{1}{s^{1.45}}$, corresponding time and frequency responses are as follows:

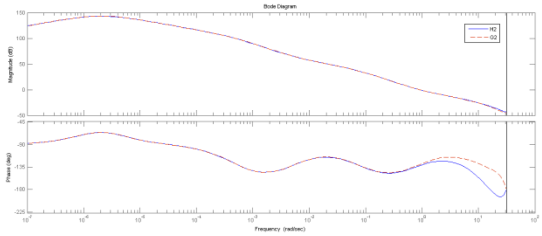

Figure 3.1 **Frequency** response of a fractional order integrator of order 1.45 using modified oustaloup approximations with zoh (H2) and foh (G2) methods

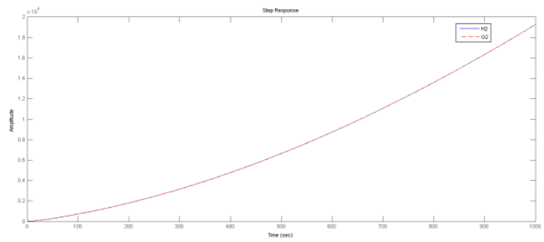

Figure 3.2 **Time** response of a fractional order integrator of order 1.45 using modified oustaloup approximations with zoh (H2) and foh (G2) methods

Using Tustin, prewarp and matched method and modified oustaloup approximation for transfer function $\frac{1}{s^{1.45}}$, corresponding time and frequency responses are as follows

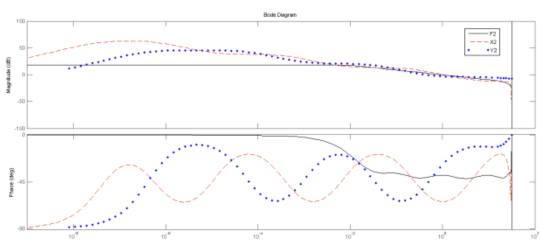

Figure 3.3 **Frequency** response of a fractional order integrator of order 1.45 using oustaloup approximations with Tustin (F2), prewarp(X2) and matched (Y2) methods

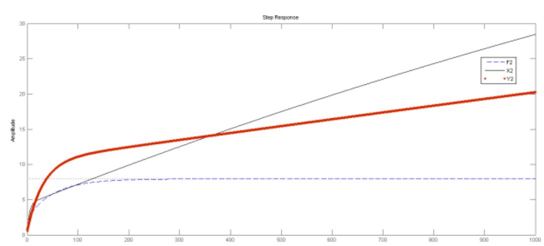

Figure 3.4 **Time** response of a fractional order integrator of order 1.45 using modified oustaloup approximations with Tustin (F2), prewarp(X2) and matched (Y2) methods.

Here for the modified oustaloup approximation methods the rise time is 650s for matched method,940s for prewarp method and 90s for tustin method.

V. CONCLUSION

From these comparative studies we can conclude that different approximation methods for the discretization of fractional order system gives different type response behavior. Also the rise time of the system varies and it is highly depended on types of filter and approximation methods. One should practically implement the fractional order equations and carry out the actual study using DSP processor to check the validity of simulation results for the different fractional order systems.

VI. REFERENCES

[1] I. Podlubny., "Fractional-order systems and controllers.," *IEEE*, p. 208–214, 1999.

[2] M. Nakagawa, K. Sorimachi, "Basic characteristics of a fractance," p. 1814–1818, 1992.

[3] A. Oustaloup, F. Levron, F. Nanot, *et al*. Oustaloup, "Frequency band complex noninteger differentiator: characterization and synthesis," *IEEE Trans. on Circuit and Systems - I: Fundamental Theory and Application, vol. 47*, pp. 25-39, 2000.

[4] B. Ross., K.S. Miller, *An Introduction to the Fractional Calculus and Fractional*. New York: John Wiley and Sons, 1993.

[5] V. Kiryakova, "Generalized Fractional Calculus and Applications," *science direct*, pp. 73-84, 1994.

[6] I. Podlubny, "The Laplace Transform Method for Linear Differential Equations of the Fractional Order," 1994.

[7] B.M. Vinagre, I. Podlubny, A. Hernandez, *et al.*, *realization of fractional-order controllers*. Lille: Conference Internationale Francophone, 2000.

[8] C. A. Monje ,Yang Quan Chen, B. M. Vinagre , Dingyu. Xue, V. Feliu, *Fractional-order Systems and Controls*. New York: Springer-Verlag London Limited, 1988.

[9] Ivo. Petr´aˇs, *Analogue Realization of Fractional Order*. Fakulta BERG, TU Koˇsice, 2002.

[10] B.M. Vinagre, I. Podlubny, A. Hern´andez, *et al* "Some approximations of fractional order operators used in control theory and applications," in , 2000, p. 231–248.

[11] B.M. Vinagre, I. Petr´aˇs, P. Merch´an, *et al.*, "Two digital realizations of fractional controllers: application to temperature control of a solid," in *European Control Conference*, Porto, Portugal, 2001, p. 1764–1767.

[12] D.Y. Xue et al, "A modified approximation method of fractional order system," in *IEEE Conference on Mechatronics and Automation*, Luoyang, China,, 2006, p. 1043–1048.

[13] K. L. Monje. Y.Q. Chen, "Discretization schemes for fractional-order differentiators and integrators.," in *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, 2002, p. 363–367.

[14] W. S. Levine, *The Control Handbook*. CRC Press and IEEE Press, 1996.