# *Stress intensity factors for cracks emanating from a circular hole in a finite orthotropic lamina*

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***Abstract:*** **The stress intensity factors (SIF) for cracks emanating from a circular hole in a finite orthotropic lamina subjected to i) uniaxial loading, ii) biaxial loading and iii) mixed boundary condition are obtained using the modified mapping collocation method. The presented analysis is based on choosing the stress functions satisfying the stress free boundary condition along the crack. The coefficients of the stress functions are determined by satisfying the boundary condition on the border of the lamina. The effect of hole radius, crack length, width of lamina, for various fibre orientations and, biaxial loading factor, mixed boundary condition for fibre orientation $90^0$ on stress intensity factors is reported.**

***Keywords-stress intensity factors (SIF), finite orthotropic lamina, modified mapping collacation***

## I. NOMENCLATURE

| | |
|---|---|
| $a_{11}$,$a_{12}$,$a_{22}$, $a_{16}$,$a_{26}$,$a_{66}$ | Elastic Constants for orthotropic lamina |
| P | Applied Stress in y-direction |
| Q | No of collocation points |
| $E_1$,$E_2$ | Modulii of Elasticity in principal material direction |
| $G_{12}$ | Shear Modulus |
| i | Imaginary number equal to √-1 |
| Re | Real part |
| $S_k$ (k=1,2) | Complex parameters of anisotropy |
| u,v | displacements in x and y direction respectively |
| $\sigma_x$,$\sigma_y$ | Stresses in x and y directions respectively |
| $\tau_{xy}$ | Shear stress in x-y plane. |
| $\upsilon_{12}$ | Poisson's Ratio |
| $\zeta$ | Mapping coordinate |
| $K_I$, $K_{II}$ | Mode I and Mode II Stress intensity Factors respectively, |
| $Y_I$, $Y_{II}$ | Normalized Mode I and Mode II Stress intensity Factors respectively |
| r,l | Hole radius and length of the crack respectively |
| 2H,2W | Height and width of the lamina respectively |
| k | Bi-axial loading factor |
| h | Fraction of the height subjected to displacement constraint. |
| N,M,K | No. of terms associated with the stress functions |
| $A_n$,$B_n$ | Unknown coefficients in the stress functions. |
| $\alpha$ | Angle of fibre orientation |
| $z_k$ | Complex coordinate, $z_k$=x+$s_k$y. |
| $\varepsilon_x$, $\varepsilon_y$, $\gamma_{xy}$ | Longitudinal, transverse and shear strains |

## II. INTRODUCTION

Owing to its importance in the fracture mechanics, the analysis of cracks emanating from a circular hole in an infinite anisotropic plate is addressed in literature by many authors. S S Wang and J F Yau [1] have developed finite element formulation for determination of mixed mode SIF and associated energy release rate. Liaw and Kamel [2] obtained SIF for cracks emanating from circular/elliptical hole and plate subjected to uniaxial loading / biaxial loading, crack subjected to internal pressure / shear. Sharma D S and Ukadgaonkar V G [3] determined stress functions by evaluating Schwarz's integral for the given boundary condition for one/two cracks emanating from a circular hole and subjected to internal pressure / shear.

Modified mapping collocation method which combines the features of conformal mapping and boundary collocation arguments is developed by Bowie and Neal [4] and extended for orthotropic materials by Bowie and Freese [5] as the procedure of analysis of internal cracks in finite geometry. Further, the method was used by Cheong and Hong [6,7] for analysis of cracks emanating from circular hole in an orthotropic plate i) under mixed mode deformations and ii) $[0_n/90_m]_s$ laminate under various boundary conditions.

The effect of biaxial loading on the SIF is studied by , Olademaji [8] for crack emanating from a circular hole in an isotropic plate, Won-Kyun Lim et al.[9] for effect on crack extension in anisotropic plates containing a central crack, Xiangqiao Yan [10] for cracks emanating from a square hole in rectangular sheets using the boundary element method.

## III. BASIC EQUATIONS

The generalized Hooke's Law in plane stress for an orthotropic lamina can be expressed in the following manner[11].

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_x \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad (1)$$

Representing $\sigma_x$,$\sigma_y$, $\tau_{xy}$ in terms of Airy's stress function F(x,y) ,



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$$\sigma_x = \frac{\partial^2 F}{\partial y^2},\ \sigma_y = \frac{\partial^2 F}{\partial x^2},\ \tau_{xy} = \frac{-\partial^2 F}{\partial x\,\partial y} \tag{2}$$

The compatibility condition is,

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} \tag{3}$$

By introducing Hooke's law in terms of F(x,y) into the compatibility equation, we get following bi-harmonic equation

$$a_{22}\frac{\partial^4 F}{\partial x^4} - 2a_{26}\frac{\partial^4 F}{\partial x^3 \partial y} + (2a_{12}+a_{66})\frac{\partial^4 F}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 F}{\partial y^3 \partial x} + a_{11}\frac{\partial^4 F}{\partial y^4} = 0 \tag{4}$$

The Airy's stress functions can be represented as,

$$F(x,y) = 2\mathrm{Re}\left[F_1(z_1) + F_2(z_2)\right] \tag{5}$$

where, $z_1=x+s_1y$ and $z_2=x+s_2y$. $F_1$ and $F_2$ are analytic functions of complex variables $z_1$ and $z_2$ respectively. The complex parameters $s_1$ and $s_2$ are roots of characteristic equation

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12}+a_{66})s^2 - 2a_{26}s + a_{22} = 0 \tag{6}$$

The stress components are represented in terms of stress functions,

$$\begin{aligned}\sigma_x &= 2\mathrm{Re}\left[s_1^2\varphi_1'(z_1) + s_2^2\varphi_2'(z_2)\right]\\ \sigma_y &= 2\mathrm{Re}\left[\varphi_1'(z_1) + \varphi_2'(z_2)\right]\\ \tau_{xy} &= -2\mathrm{Re}\left[s_1\varphi_1'(z_1) + s_2\varphi_2'(z_2)\right]\\ \text{where, } \varphi_1(z_1) &= F_1'(z_1) \text{ and } \varphi_2(z_2) = F_2'(z_2)\end{aligned} \tag{7}$$

The boundary condition of traction type is expressed as,

$$f_1(s) + if_2(s) = (1+is_1)\varphi_1(z_1) + (1+is_2)\varphi_2(z_2) + (1+i\bar{s}_1)\overline{\varphi_1(z_1)} + (1+i\bar{s}_2)\overline{\varphi_2(z_2)} + c \tag{8}$$

The displacement components u and v are,

$$\begin{aligned}u &= 2\mathrm{Re}\left[p_1\varphi_1(z_1) + p_2\varphi_2(z_2)\right]\\ v &= 2\mathrm{Re}\left[q_1\varphi_1(z_1) + q_2\varphi_2(z_2)\right]\end{aligned} \tag{9}$$

Where,

$$p_k = a_{11}s_k^2 + a_{12} - a_{16}s_k,\ q_k = \frac{a_{12}s_k^2 + a_{22} - a_{26}s_k}{s_k} \quad k = (1,2)$$

## IV. MATHEMATICAL FORMULATION

Consider the straight crack emanating from the circular hole in a finite orthotropic lamina as shown in Fig. 1.
The mapping function used is,

$$z = \omega(\zeta) = \frac{L}{2}\left(\frac{1}{\zeta} + \zeta\right) \tag{10}$$

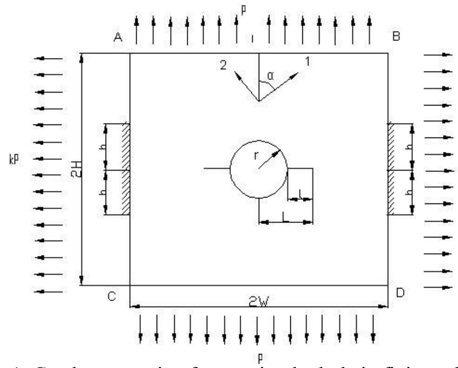


Fig. 1 Cracks emanating from a circular hole in finite orthotropic lamina with various loading conditions.

This mapping function carries the unit circle and its exterior in the ζ- plane into the crack and its exterior. The outer boundaries correspond to a closed counter in the ζ-plane exterior to the unit circle with coordinate points,

$$\zeta_k = \frac{z_k}{L} + \left[\left(\frac{z_k}{L}\right)^2 - 1\right]^{\frac{1}{2}} \quad (k = 1,2) \tag{11}$$

Let $S_{\zeta 1}^{+}$, $S_{\zeta 2}^{+}$ denote two parameter regions corresponding to $\zeta_1$ and $\zeta_2$ respectively and their union be denoted by $S_{\zeta}^{+}$. The region $S_{\zeta}^{+}$ is bounded from within by unit circle in the ζ -plane The region inside the unit circle is denoted by $S_{\zeta}^{-}$.In order to select the stress function analytic in $S_{\zeta}^{+}$ and extend it by definition into $S_{\zeta}^{-}$ following relationships can be introduced

$$\varphi_2(\zeta_2) = B\overline{\varphi_1}\left(\frac{1}{\zeta_2}\right) + C\varphi_1(\zeta_2) \quad \text{where, } \overline{\varphi_1}\left(\frac{1}{\zeta}\right) = \overline{\varphi_1\left(\frac{1}{\bar{\zeta}}\right)} \tag{12}$$

$$B = \frac{(\bar{s}_2 - \bar{s}_1)}{(s_2 - \bar{s}_2)},\ C = \frac{(\bar{s}_2 - s_1)}{(s_2 - \bar{s}_2)}$$

The determination of stress function requires a form of representation that satisfies traction free condition in the parameter plane corresponding to hole and crack in physical plane [7].

$$\varphi_1(\zeta) = \sum_{n=-N}^{N} A_n\zeta^{2n+1} + \sum_{n=1}^{K} B_n\zeta(\zeta^2+1)^{-n} \tag{13}$$

$A_n$ and $B_n$ are coefficients.

For convenience in notation we denote

$$\begin{aligned}\varphi_k(z_k) &= \varphi_k[\omega(\zeta_k)] = \varphi_k(\zeta_k),\ \varphi'_k(z_k) = \frac{\varphi'_k(\zeta_k)}{\omega'(\zeta_k)} \quad (k = 1,2)\\ \omega'(\zeta_k) &= \frac{L}{2}\left(1 - \frac{1}{\zeta_k^2}\right) \quad (k = 1,2)\end{aligned} \tag{14}$$

After considering (14), stress components, resultant forces and displacements can be expressed,

$$\sigma_x = 2\mathrm{Re}\left[s_1^2\frac{\varphi'_1(\zeta_1)}{\omega'(\zeta_1)} + s_2^2\frac{\varphi'_2(\zeta_2)}{\omega'(\zeta_2)}\right]$$
$$\sigma_y = 2\mathrm{Re}\left[\frac{\varphi'_1(\zeta_1)}{\omega'(\zeta_1)} + \frac{\varphi'_2(\zeta_2)}{\omega'(\zeta_2)}\right]$$
$$\tau_{xy} = -2\mathrm{Re}\left[s_1\frac{\varphi'_1(\zeta_1)}{\omega'(\zeta_1)} + s_2\frac{\varphi'_2(\zeta_2)}{\omega'(\zeta_2)}\right] \tag{15}$$

$$f_1(s) + i f_2(s) = (1 + i s_1)\varphi_1(\zeta_1) + (1 + i s_2)\varphi_2(\zeta_2) + (1 + i\bar{s}_1)\overline{\varphi_1(\zeta_1)} + (1 + i\bar{s}_2)\overline{\varphi_2(\zeta_2)} + c \tag{16}$$

$$u = 2\mathrm{Re}\left[p_1\varphi_1(\zeta_1) + p_2\varphi_2(\zeta_2)\right]$$
$$v = 2\mathrm{Re}\left[q_1\varphi_1(\zeta_1) + q_2\varphi_2(\zeta_2)\right] \tag{17}$$

The SIF are determined from the stress functions as follows [12]

$$K_I = 2\sqrt{2\pi}\left[\frac{s_2 - s_1}{s_2}\right]\lim_{z_1\to z_0}(z_1 - z_0)^{\frac{1}{2}}\,\varphi'_1(z_1)$$
$$K_{II} = 2\sqrt{2\pi}\left[s_2 - s_1\right]\lim_{z_1\to z_0}(z_1 - z_0)^{\frac{1}{2}}\,\varphi'_1(z_1) \tag{18}$$

Alternaively,

$$K_I = 2\sqrt{2\pi}\left[\frac{s_2 - s_1}{s_2}\right]\lim_{z_2\to z_0}(z_2 - z_0)^{\frac{1}{2}}\,\varphi'_2(z_2)$$
$$K_{II} = 2\sqrt{2\pi}\left[s_2 - s_1\right]\lim_{z_2\to z_0}(z_2 - z_0)^{\frac{1}{2}}\,\varphi'_2(z_2) \tag{19}$$

$z_0$ is the crack tip and $z_0$=L in the present case.
The results obtained as per (18) are verified with those obtained by (19).

## V. NUMERICAL RESULTS AND DISCUSSION

### A. Solution Procedure

Completion of the solution requires determination of coefficients $A_n$ and $B_n$ in the stress functions such that prescribed boundary conditions of stress components, resultant forces and displacements (only in case of mixed boundary problem) are met. To achieve this, a finite number of boundary stations were chosen in the z-plane which, in turn along with the material properties, determine $\zeta_1$ and $\zeta_2$. The expression for error with respect to stress components, resultant forces and displacements at each point are obtained. The coefficients $A_n$ and $B_n$ are then determined by using least square method introduced by Newman [13]. The system of equations are solved using Mathcad.

### B. Convergence

In order to analyze the convergence of method for present stress functions, analysis is carried out for various parameters in (13) viz. number of negative terms M, number of positive terms N, number of positive terms K, and number of collocation points, Q. In the calculation, H/W=1, r=0.2 W, L=0.28W, Glass/Epoxy with $\alpha=90^0$ are chosen. One parameter is changed at a time and it is observed that with M= 23 N=23 K=21 Q=80 the SIF obtained are satisfactory and are shown in Table I.

### C. Comparision

The comparison some of results obtained by the present method is made with those obtained by i) Wang et al. [14] for uniaxial loading using boundary collocation method and ii) Oldemaji [8] for biaxial loading with k=-2. In case of isotropic plates, comparison is also made with results obtained with finite element analysis using ANSYS 10.0 using the PLANE 82 elements. The material properties of Glass/Epoxy used in the present analysis are as follows.

$E_1$=53.74 GPa, $E_2$=17.91 GPa, $G_{12}$=8.96 GPa, $\upsilon_{12}$=0.25

TABLE I COMPARISION OF RESULTS

| | | Wang et al. [14]* | Oldemaji [8].** | Present | ANSYS |
|---|---|---|---|---|---|
| Isotropic $Y_I$ | | 2.308 | 2.000 | 2.352*<br>1.9554** | 2.414*<br>2.1064** |
| Lamina [90] $Y_I$ | | 2.102 | -- | 2.142 | -- |
| Lamina [45] | $Y_I$ | 0.887 | -- | 0.886 | -- |
| | $Y_{II}$ | 0.12 | | 0.123 | |

* H/W=1, r=0.2 W, L=0.28W
**H/W=2.5,L/W=0.5,l/r=3 k=-2
$Y_I=K_I/P(\pi L)^{1/2}$ $Y_{II}=K_{II}/P(\pi L)^{1/2}$

### D. SIF FOR VARIOUS LOADING CONDITIONS

**Case-I** -Uniaxial loading i.e. k=0, lateral edges AC and BD free of any displacement constraint and $\sigma_y$=P along AB and CD, H/W=1, r/W=0.2. The normalized SIF $Y_I=K_I/P(\pi L)^{1/2}$ $Y_{II}=K_{II}/P(\pi L)^{1/2}$ for various fibre angles viz. $0^0,30^0,45^0,60^0,90^0$ are plotted in Fig. 2 and Fig. 3. It is noted that there exists a marginal difference between $Y_{II}$ values for $\alpha=30^0$ and $\alpha=45^0$.Also, i) for a given L/W ratio $Y_I$ for $\alpha=90^0$ is approximately five times of that for $\alpha=0^0$ ii)for a given fibre orientation, the $Y_I$ and $Y_{II}$ nearly doubles as L/W increases from 0.2 to 0.8. Fig. 4 shows $Y_I$ for various l/r and L/W ratios for $\alpha=90^0$ . SIF L/W=1/3 and L/W=1/6 are approximately same for all l/r values. $Y_I$ approach a constant value when l/r>2 before reaching maximum for around l/r=0.25 indicating, this geometry configuration be avoided for engineering applications.

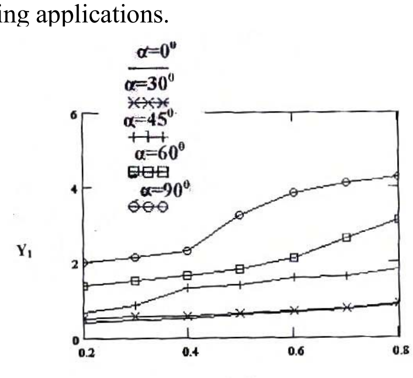


Fig. 2. Normalized SIF Mode I Vs. L/W for various fibre orientations.

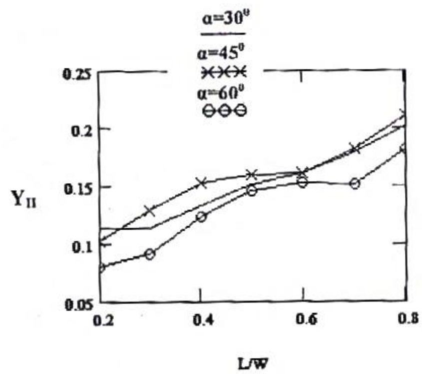


Fig. 3. Normalized SIF Mode II Vs L/W for various fibre orientations.

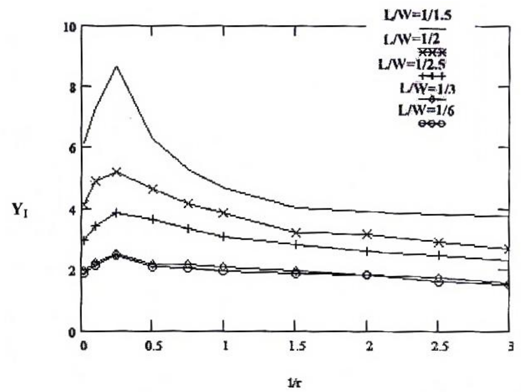


Fig. 4. Normalized Mode I SIF for various l/r and L/W ratios for α=90$^0$

**Case-II** – Biaxial loading, $\sigma_y$=P along AB and CD and $\sigma_x$=kP along AC and BD with AC and BD being free of displacement constraint, H/W=1, r/W=0.2. The variation of $Y_I$ with L/W for various biaxial loading factors k=-2,k=-1,k=0,k=2,k=1 and for α=90$^0$ is shown in Fig. 5 indicating that difference in $Y_I$ with respect to k=0 decreases with increasing L/W ratio.

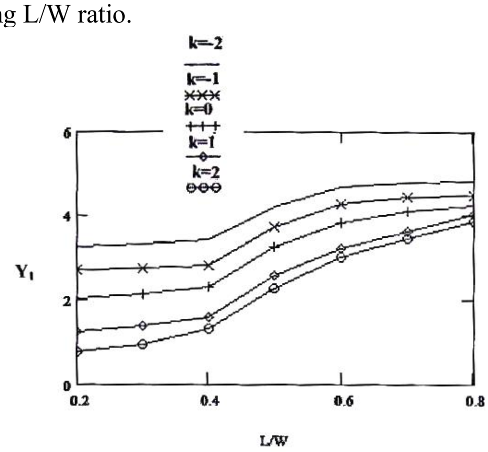


Fig. 5 Effect of biaxial loading on SIF for α=90$^0$

**Case-III**- Mixed Boundary condition with $\sigma_y$=P along AB and CD, k=0 and lateral edges AC and BD subjected to displacement constraint for the portion “2h”, H/W=1, r/W=0.2 and α=90$^0$ . It is clear from Fig. 6 that constraining has no effect on decreasing $Y_I$ if L/W<0.4 but considerable effect on decreasing $Y_I$ for larger L/W and h/H ratios.

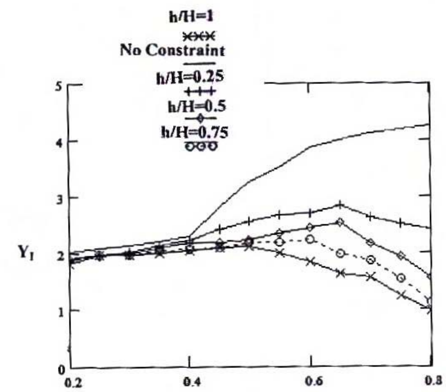


Fig. 6. Effect of constraining the lateral edges on SIF for α=90$^0$

## VI. CONCLUSION

The SIF for finite orthotropic lamina with cracks emanating from circular hole are obtained for uniaxial loading, biaxial loading and mixed boundary condition using modified mapping collocation method. The solution takes care of fibre orientation, hole radius, crack length and plate width. The convergence of the method for present stress functions is analyzed. The results are compared with the available literature and those obtained using finite element analysis. The method can be extended further to analysis of i) finite orthotropic lamina subjected to combination of uniaxial loading and shear ii) effect of stacking sequence in a laminate on SIF.

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