Stress Distribution Around Triangular Hole in Orthotropic Plate

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***Abstract*--General solutions for determining the stress field around triangular hole in infinite orthotropic plate subjected to in-plane loading are obtained using Muskhelisvili's complex variable formulation. The generalized formulation thus obtained is coded and few numerical results are obtained using MATLAB 7.6. The effect of loading factor, corner radius, fibre orientation and material parameter on stress pattern around triangular hole is studied. Some of the results are compared with the results available from the literature.**

***Index terms*—Loading factor, orthotropic plate, stress function, Triangular hole.**

I. INTRODUCTION

Many researchers have tried to find stress fields around holes and cutouts in isotropic /anisotropic media. The stress distribution around triangular hole in orthotropic plate is not studied extensively and there is scope for research.

Ukadgaonker and Rao[1], and Daoust and Hoa[2] found stress patterns around triangular hole using Muskhelisvili's [3] complex variable approach. Ukadgaonker and Rao[1] adopted Gao's[4] biaxial loading factor to apply arbitrary biaxial loading conditions at infinity. Daoust and Hoa[2] studied effect of various length to height ratio of triangle, degrees of bluntness and orientation of load on the stress pattern.

Using Muskhelisvili's [3] complex variable approach the generalized solution for stress distribution around triangular hole in an infinite orthotropic plate under uni-axial, biaxial and shear loading is obtained. In order to consider several cases of in-plane loads, the arbitrary biaxial loading condition is introduced. The generalized solutions obtained are coded in MATLAB 7.6 and the effect of loading condition, corner radius and material property on stress pattern is studied. The results are compared with the existing literature.

II. COMPLEX VARIABLE FORMULATION

A thin anisotropic plate is considered under generalized plane stress condition (Figure 1). The plate is assumed to be loaded in such a way that resultants lies in XOY plane. The stresses on top and bottom surface of plate as well as σ_z, τ_{xz} and τ_{yz} are zero everywhere within the plate. Using generalized Hooke's law, Airy's stress function and strain-displacement compatibility condition, the following characteristic equation is obtained, roots of which represents constant of anisotropy.

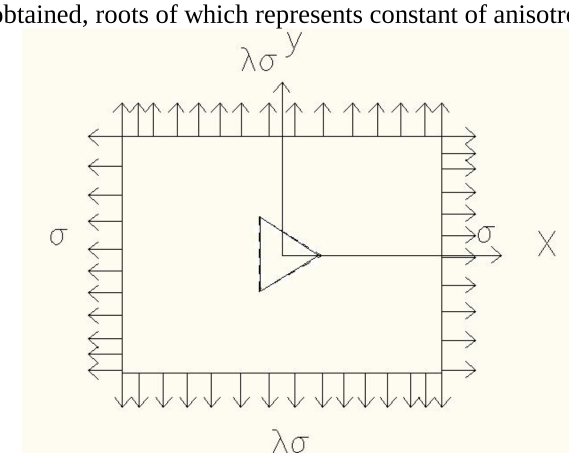


Fig.1 Plate with triangular hole

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0$$

a_{ij} are the compliance co-efficients.

The roots of this equation are;

$$s_1 = \alpha_1 + i\beta_1;\ s_2 = \alpha_2 + i\beta_2;$$
$$s_3 = \alpha_1 - i\beta_1;\ s_4 = \alpha_2 + i\beta_2 \quad \text{…..(1)}$$

The Airy's stress function U(x,y) can be represented as

$$U(x,y) = F_1(x + s_1 y) + F_2(x + s_2 y) + F_3(x + s_3 y) + F_4(x + s_4 y) \quad \text{..…(2)}$$

$$U(x,y) = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)}$$

The analytic functions $\phi(z_1), \psi(z_2)$ and their conjugates are given below.

$$\frac{dF_1}{dz_1} = \phi(z_1), \qquad \frac{dF_2}{dz_2} = \psi(z_2),$$
$$\frac{d\overline{F_1}}{d\overline{z_1}} = \overline{\phi(z_1)}, \qquad \frac{d\overline{F_2}}{d\overline{z_2}} = \overline{\psi(z_2)} \quad \text{….(3)}$$

$\phi(z_1)$ and $\psi(z_2)$ are the Mushkhelishvili's complex function. The stress components for plane stress conditions can be written in terms of these stress functions as follows:

$$\sigma_x = 2\,\mathrm{Re}\left[s_1^{\,2}\phi'(z_1) + s_2^{\,2}\psi'(z_2)\right]$$
$$\sigma_y = 2\,\mathrm{Re}\left[\phi'(z_1) + \psi'(z_2)\right]$$

$$\tau_{xy} = -2\,\mathrm{Re}\left[s_1\phi'(z_1) + s_2\psi'(z_2)\right] \qquad \text{....(4)}$$

The stresses in Cartesian coordinates given in equation (4) can be written in orthogonal curvilinear coordinate system by means of the following relations

$$\sigma_\theta + \sigma_\rho = \sigma_x + \sigma_y$$

$$\sigma_\theta - \sigma_\rho + 2i\tau_{\rho\theta} = (\sigma_y - \sigma_x + 2i\tau_{xy})e^{2i\theta} \qquad \text{....(5)}$$

The area external to a given triangular hole, in Z-plane is mapped conformably to the area outside the unit circle in ζ plane using following mapping function.

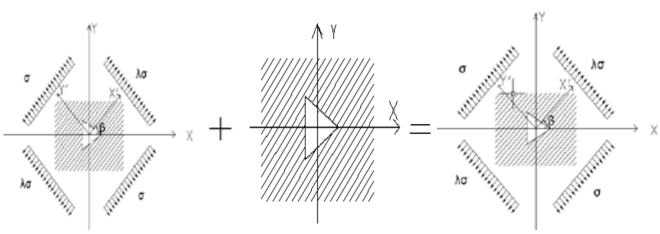
$$z_j = \omega_j(\xi) = \frac{R}{2}\left[a_j\left(\frac{1}{\xi} + \sum_{k=1}^{N} m_k \xi^k\right) + b_j\left(\xi + \sum_{k=1}^{N}\frac{m_k}{\xi^k}\right)\right]$$

Where k=1, 3, 5, 8,11,14,17

$$a_j = (1 + is_j), \qquad b_j = (1 - is_j)\ ; \quad j=1,2. \qquad \text{...(6)}$$

Gao's[4] arbitrary biaxial loading condition is adopted to facilitate solution of plate subjected to biaxial loading. By introducing biaxial loading factor, λ and loading angle, α, method of superposition of solutions of two uni-axially loaded plate can be avoided.

III. STRESS FUNCTION FOR TRIANGULAR HOLE PROBLEM



(1) (2) (3)

Fig.2.Problem configuration with scheme of solution

The scheme for solution of orthotropic plate containing a triangular hole subjected to remotely applied load is shown in Figure 2. To determine the stress function, the solution is split into two stages:

1) *First Stage*

The stress functions $\phi(z_1)$ and $\psi(z_2)$ are determined for the hole free plate under the application of remotely applied load. The boundary conditions f_1 and f_2 are found for the fictitious hole using stress functions $\phi(z_1)$ and $\psi(z_2)$.

The stress function $\phi(z_1)$ and $\psi(z_2)$ are obtained for hole free plate due to remotely applied load $\sigma_x^{\infty}, \sigma_y^{\infty}$.

$$\phi_1(z_1) = B^* z_1$$

$$\psi_1(z_2) = (B'^* + iC'^*)z_2 \qquad \text{...(7)}$$

Where,

$$B^* = \frac{\sigma_x^{\infty} + (\alpha_2^{\,2} + \beta_2^{\,2})\sigma_y^{\infty} + 2\alpha_2\tau_{xy}^{\infty}}{2((\alpha_2 - \alpha_1)^2 + (\beta_2^{\,2} - \beta_1^{\,2}))}$$

$$B'^* = \frac{(\alpha_1^{\,2} - \beta_1^{\,2} - 2\alpha_1\alpha_2)\sigma_y^{\infty} - \sigma_x^{\infty} - 2\alpha_2\tau_{xy}^{\infty}}{2((\alpha_2 - \alpha_1)^2 + (\beta_2^{\,2} - \beta_1^{\,2}))}$$

$$C'^* = \left[\frac{[(\alpha_1 - \alpha_2)]\sigma_x^{\infty} + [\alpha_2(\alpha_1^{\,2} - \beta_1^{\,2}) - \alpha_1(\alpha_2^{\,2} - \beta_2^{\,2})]\sigma_y^{\infty} + [(\alpha_1^{\,2} - \beta_1^{\,2}) - (\alpha_2^{\,2} - \beta_2^{\,2})]\tau_{xy}^{\infty}}{2\beta_2[(\alpha_2 - \alpha_1)^2 + (\beta_2^{\,2} - \beta_1^{\,2})]}\right]. \qquad \text{...(8)}$$

C is taken zero, because no rotation is allowed.

The boundary conditions f_1, f_2 on the fictitious hole are determined from these stress functions as follows.

$$f_1 = \left[(K_1 + \overline{K_2})\left(\frac{1}{\xi} + \sum_{k=1}^{N} m_k\xi^k\right) + (K_2 + \overline{K_1})\left(\xi + \sum_{k=1}^{N}\frac{m_k}{\xi^k}\right)\right]$$

$$f_2 = \left[(K_3 + \overline{K_4})\left(\frac{1}{\xi} + \sum_{k=1}^{N} m_k\xi^k\right) + (K_4 + \overline{K_3})\left(\xi + \sum_{k=1}^{N}\frac{m_k}{\xi^k}\right)\right] \qquad \text{...(9)}$$

Where,

$$K_1 = \left(\frac{R}{2}\right)\left[B^* a_1 + (B'^* + iC'^*)a_2\right]$$

$$K_2 = \left(\frac{R}{2}\right)\left[B^* b_1 + (B'^* + iC'^*)b_2\right]$$

$$K_3 = \left(\frac{R}{2}\right)\left[s_1 B^* a_1 + s_2(B'^* + iC'^*)a_2\right]$$

$$K_4 = \left(\frac{R}{2}\right)\left[s_1 B^* b_1 + s_2(B'^* + iC'^*)b_2\right]$$

2) *Second Stage*

For the second stage solution, the stress functions $\phi_0(z_1)$ and $\psi_0(z_2)$ are determined by applying negative of the boundary conditions $f_1^0 = -f_1$ and $f_2^0 = -f_2$ on its hole boundary in the absence of the remote loading.

The stress functions of second stage solution are obtained using these new boundary conditions (f_1^0, f_2^0) into Schwarz formula:

$$\psi_0(\xi) = \frac{i}{4\Pi(s_1 - s_2)} \int_\gamma \left[(s_1 f_1^0 - f_2^0) \left\{ \frac{t+\xi}{t-\xi} \right\} \frac{dt}{t} \right]$$
$$\phi_0(\xi) = \frac{i}{4\Pi(s_1 - s_2)} \int_\gamma \left[(s_2 f_1^0 - f_2^0) \left\{ \frac{t+\xi}{t-\xi} \right\} \frac{dt}{t} \right] \quad \ldots(10)$$

By evaluating the integral the stress functions are obtained as

$$\phi_0(\xi) = \left\{ \frac{a_3}{\xi} + b_3 \sum_{k=1}^{N} \frac{m_k}{\xi^k} \right\}$$
$$\psi_0(\xi) = -\left\{ \frac{a_4}{\xi} + b_4 \sum_{k=1}^{N} \frac{m_k}{\xi^k} \right\} \quad \ldots(11)$$

Where,

$$a_3 = \left\{ \frac{1}{s_1 - s_2} \right\} \left[s_2 (K_1 + \overline{K_2}) - (K_3 + \overline{K_4}) \right]$$
$$b_3 = \left\{ \frac{1}{s_1 - s_2} \right\} \left[s_2 (K_2 + \overline{K_1}) - (K_4 + \overline{K_3}) \right]$$
$$a_4 = \left\{ \frac{1}{s_1 - s_2} \right\} \left[s_1 (K_1 + \overline{K_2}) - (K_3 + \overline{K_4}) \right]$$
$$b_4 = \left\{ \frac{1}{s_1 - s_2} \right\} \left[s_1 (K_2 + \overline{K_1}) - (K_4 + \overline{K_3}) \right]$$

3) *Final Solution*

The stress function $\phi(z_1)$ and $\psi(z_2)$ for single hole problem, can be obtained by adding the stress functions of first and second stage.

$$\phi(z_1) = \phi_1(z_1) + \phi_0(z_1)$$
$$\psi(z_2) = \psi_1(z_2) + \psi_0(z_2) \quad \ldots(12)$$

IV. NUMERICAL SOLUTION OF THE BASIC FORMULATION

1. Choose the value of biaxial load factor, λ and load angle, α for the type of loading.
2. Calculate the compliance co-efficient, a_{ij} from generalized Hooke's Law.
3. Calculate the value of complex parameters of anisotropy s_1 and s_2 from the characteristic equation.
4. Calculate the constants: $a_1, b_1, a_2, b_2, B, B', C', K_1, K_2, K_3, K_4$ etc.
5. Evaluate the stress functions and their derivatives.
6. Evaluate stresses.

V. RESULT AND DISCUSSION:

The stress functions obtained above are the generalized solutions. Using these functions, stress distribution for different loading conditions and material parameters can be obtained.

The following loading cases have been considered.

1. Plate subjected to uni-axial tension at infinite distance.
2. Plate subjected to biaxial tension at infinite distance.
3. Plate subjected to shear at infinite distance.

The mapping function having 7 terms is used. As number of terms increases the hole shape converges to equilateral triangle and corner radius decreases. This convergence can be seen from Figure 3.

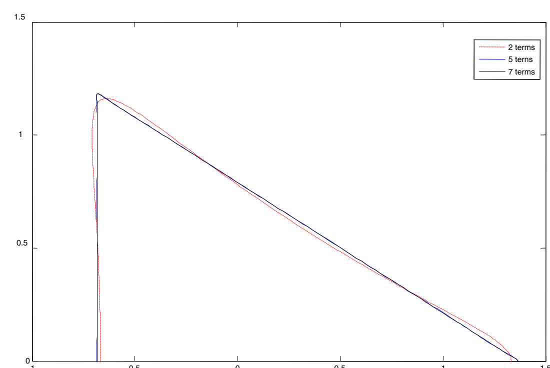


Fig 3. Shape for equilateral triangle with different degree of bluntness at the apex.

The material properties used for numerical solution and constants of anisotropy are tabulated below (Table I).

TABLE I.
MATERIAL PROPERTIES FOR DIFFERENT MATERIAL

Material	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	ν_{12}
Isotropic steel	207	207	79.3	0.3

CE 9000 Glass/Epoxy	47.4	16.2	7	0.26
Plywood	11.79	5.89	0.69	0.071
T300/5208 Graphite/Epoxy	181	10.3	7.17	0.28

The two term solution for Graphite/Epoxy, Isotropic steel, Plywood and Glass/Epoxy is shown in Figure 4. The normalized stresses are compared with Daoust and Hoa[2] and found in close agreement. The mapping function, material parameters and loading condition are taken same as Daoust and Hoa[2] for sake of comparison of numerical results. Actually, the solution presented here is capable of handling any orthotropic/isotropic material and in-plane loading at infinity. The comparisons of maximum normalized tangential stress are shown in Table II.

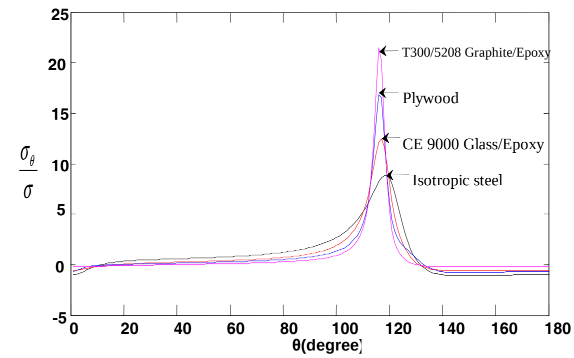


Fig 4.Stress distribution around hole for different material

TABLE II.
THE CONSTANTS OF ANISOTROPY (S_1, S_2) AND COMPARISONS OF MAXIMUM NORMALIZED TANGENTIAL STRESS

Material	Constants of anisotropy (s_1, s_2)	Present method $\frac{\sigma_\theta}{\sigma}$	Daoust and Hoa[2] $\frac{\sigma_\theta}{\sigma}$
Isotropic steel	s_1=i s_2=i	8.4906	8.28
CE 9000 glass/epoxy	s_1=2.3962i s_2=0.7138i	12.4846	12.13
Plywood	s_1=4.1019i s_2=0.3449i	16.86	17.04
T300/5208 graphite/epoxy	s_1=4.8939i s_2=0.8566i	21.48	21.63

The stress field around triangular hole in isotropic steel plate subjected to uni-axial and biaxial loading can be seen from Figure 5. For uni-axial loading, loading angle $\alpha=0^0$, the maximum normalized tangential stress is found 11.0129 at $\theta=0^0$ and for loading angle $\alpha=90^0$, maximum normalized tangential stress is obtained 8.8537 at $\theta=120^0$. For biaxial loading, the maximum normalized tangential stress is found 10.0129 at 0^0, 120^0 and 240^0.

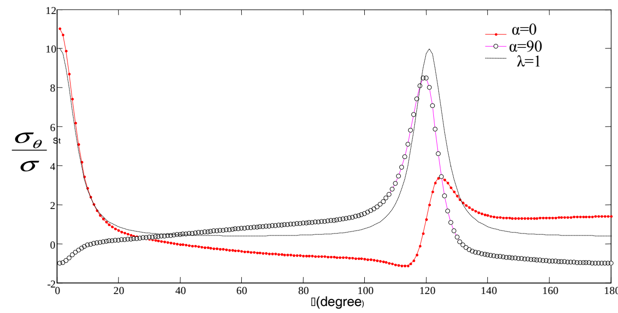


Fig.5 Effect of loading on stress concentration in isotropic steel plates

For the same fiber orientation angle $\beta=0^0$, the effect of loading factor on stress field for Graphite/Epoxy can be seen from Figure 6. The maximum normalized stresses for uni-axial load at 0^0, at 90^0, biaxial load and shear load at infinity are found 7.8589(at 0^0), 21.4828(at 120^0), 20.2330(at 120^0) and 21.3419(at 240^0) respectively.

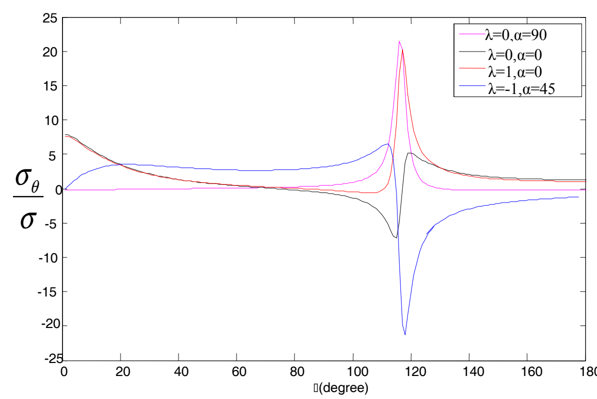


Fig.6 Effect of loading on stress concentration in Graphite/Epoxy plates

When glass/epoxy infinite plate with fiber orientation angle $\beta=0^0$ and $\beta=90^0$ is subjected to equi-biaxial loading, the maximum stress concentration factor is found 12.9780 and 15.088, respectively (Refer Figure 7). For fiber orientation angle $\beta=0^0$, the maximum tangential stress is found at 120^0, while for fiber orientation angle $\beta=90^0$, the maximum tangential stress found at 0^0.

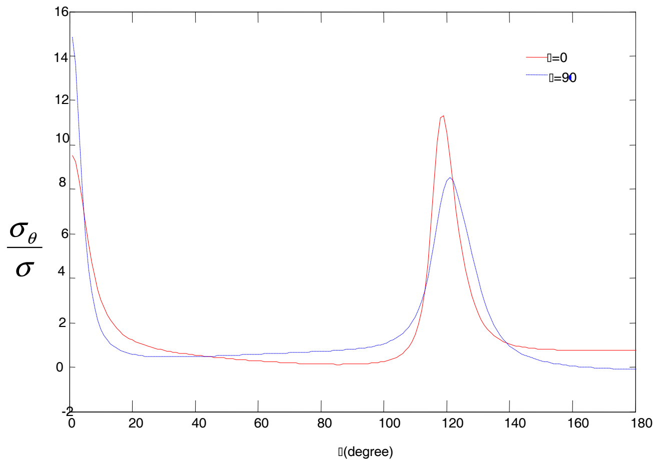


Fig.7 Effect of fiber angle on stress concentration in glass/epoxy plates

VI. CONCLUSION

The generalized solutions have been obtained for two dimensional stress distribution around equilateral triangular hole for orthotropic /isotropic plate under tensile load in any direction and shear load. The effect of corner radius, fiber orientation, loading angle and material property on stress distribution is studied. For isotropic material as number of terms in the mapping function increases (Corner radius decreases) the stress concentration at the corners increases. The stress concentration is different for different orthotropic materials for the same loading and hole geometry.

VII. REFERENCES

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