Stress Analysis of Infinite Isotropic Plate with a internally pressurized square Hole Subjected to In-plane loading at Infinity

Dr D S Sharma[1], Patel Nirav[2]
1,2 Institute of Technology, Nirma University, Ahmedabad

***Abstract*--General solution for determining stress concentration around square hole in infinite isotropic plate subjected to in plane tension/shear at infinity and internal pressure and/or shear on hole boundry are obtained using complex variable method . For obtaining stress functions, Cauchy's integral is evaluated for given boundary conditions. The generalize formulation is coded using MATLAB 6.5 and stress field around the elliptical opening is obtain. Some of the results are compared with finite element results using ANSYS.**

***Index Terms*-- stress function, internal pressure, infinite plate**

I. Introduction

IN practice depending on the application and different service requirements holes are made in structures and machines. These holes work as stress raisers and hence it is desirable to know stress distribution around them.

Kolosov-Muskhellishvilli's[1] complex variable approach is useful and handy tool to study two dimensional stress analysis problem. Here, using this approach generalized solution for finding stress distribution around a square hole is presented. Gao's[2] arbitrary biaxial loading condition is employed to facilitate the solution of bidirectional loading. The solution is capable of handling different hole orientation and loading pattern. For sake of comparison few loading cases for the square hole are solved using finite element method and the results are found in close agreement.

II. Boundary Conditions at Infinity

The boundary conditions for different in-plane loading conditions are as follows:

$$\sigma_{x'}^{\infty} = \lambda\sigma; \quad \sigma_{y'}^{\infty} = \sigma; \quad \tau_{xy'}^{\infty} = 0 \quad \text{at } |z| \to \infty \quad (1)$$

Where, $\sigma_{x'}^{\infty};\sigma_{y'}^{\infty}$; stresses applied about x`, y` axes at infinity. λ=0 and λ=1 explains uniaxial and biaxial loading conditions, respectively. The boundary conditions about XOY can be written explicitly as:

$$\sigma_x = \frac{\sigma}{2}[(\lambda+1)+(\lambda-1)\cos 2\alpha]$$
$$\sigma_y = \frac{\sigma}{2}[(\lambda+1)-(\lambda-1)\cos 2\alpha] \quad (2)$$
$$\tau_{xy} = \frac{\sigma}{2}[(\lambda-1)\sin 2\alpha]$$

III. Complex Variable Formulation

The basic equations of plane elasticity in complex variable form are given by Kolosov-Muskhellishvilli [1] as follows.

$$\sigma_x + \sigma_y = 2\left[\phi'(z) + \overline{\phi'(z)}\right] = 4\,\mathrm{Re}[\phi'(z)]$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right] \quad (3)$$

Where $\phi(z),\psi(z)$ =complex potentials of the complex variable $z = x + iy$

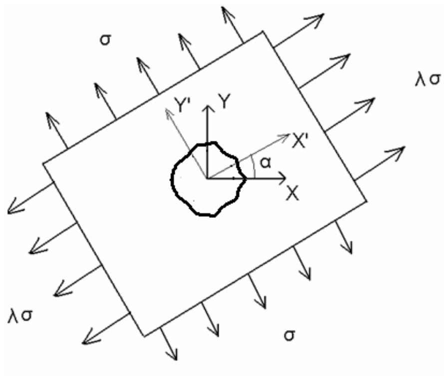

Fig 1 Arbitrary biaxial loading

In order to find stress distribution around a hole in z-plane, it is mapped to a region outside unit circle in ζ-plane, which has origin at ζ=0. The mapping function is given as follows.[3]

$$z = \omega(\zeta) = \frac{R}{2}\left(\zeta + \frac{m_3}{\zeta^3} + \frac{m_7}{\zeta^7} + \frac{m_{11}}{\zeta^{11}} + \frac{m_{15}}{\zeta^{15}} + \frac{m_{19}}{\zeta^{19}}\right) \quad (4)$$

Where m_3=-1/6, m_7=1/56, m_{11}=-1/176, m_{15}=1/384, m_{19}=-7/4864

The stress functions in ζ-plane are given by

$$\phi(\zeta) = [\phi_1(\zeta) + \phi_o(\zeta)]$$
$$\psi(\zeta) = [\psi_1(\zeta) + \psi_o(\zeta)] \quad (5)$$

In absence of hole stress functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ can be written as $\phi_1(\zeta)=\left[(B+iC)\omega(\zeta)\right]$

$$\psi_1(\zeta)=\left[(B'+iC')\omega(\zeta)\right] \quad (6)$$

Where

$$B=\frac{\sigma_x+\sigma_y}{4};\ B'=\sigma_y-\sigma_x;\ C'=\tau_{xy};C=0 \quad (7)$$

The boundary condition in ζ-plane is given by

$$f(t)=\phi(t)+\frac{\omega(t)}{\overline{\omega'(t)}}\overline{\phi'(t)}+\overline{\psi(t)} \quad (8)$$

Substituting the functions in equation 8 we get

$$f(t)=\frac{R}{2}\left[\begin{array}{l}2B\left(t+\frac{m_3}{t^3}+\frac{m_7}{t^7}+\frac{m_{11}}{t^{11}}+\frac{m_{15}}{t^{15}}+\frac{m_{19}}{t^{19}}\right)\\+(B'-iC')\\\left(\frac{1}{t}+m_3t^3+m_7t^7+m_{11}t^{11}+m_{15}t^{15}+m_{19}t^{19}\right)\end{array}\right] \quad (9)$$

When internal shear or internal pressure on hole boundary is considered the boundary condition is given as below.

$$f(t)=\frac{R}{2}\left[\begin{array}{l}2(B+p+iT)\left(t+\frac{m_3}{t^3}+\frac{m_7}{t^7}+\frac{m_{11}}{t^{11}}+\frac{m_{15}}{t^{15}}+\frac{m_{19}}{t^{19}}\right)\\+(B'-iC')\\\left(\frac{1}{t}+m_3t^3+m_7t^7+m_{11}t^{11}+m_{15}t^{15}+m_{19}t^{19}\right)\end{array}\right] \quad (10)$$

Where, p=internal pressure, T=internal shear.

by evaluating Cauchy's integral, stress function $\phi_0(\zeta)$ and $\psi_0(\zeta)$ can be obtained as follows.

$$\phi_0(\zeta)=-\frac{1}{2\pi i}\oint\frac{f(t)dt}{t-\zeta}$$

$$\psi_0(\zeta)=-\frac{1}{2\pi i}\oint\frac{\overline{f(t)}dt}{t-\zeta}-\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\phi_0'(\zeta) \quad (11)$$

Substituting equation 10 in to 11 and solving equation we get,

$$\phi_0(\zeta)=-\frac{R}{2}\left(\begin{array}{l}2(B+p+iT)\left(\zeta+\frac{m_3}{\zeta^3}+\frac{m_7}{\zeta^7}+\frac{m_{11}}{\zeta^{11}}+\frac{m_{15}}{\zeta^{15}}+\frac{m_{19}}{\zeta^{19}}\right)\\+\frac{(B'-iC')}{\zeta}\end{array}\right)$$

$$\psi_0(\zeta)=-\frac{R}{2}\left[\begin{array}{l}\left(\frac{2(B+p+iT)\left(\frac{1}{\zeta}\right)+B'+iC'}{\left(\frac{m_3}{\zeta^3}+\frac{m_7}{\zeta^7}+\frac{m_{11}}{\zeta^{11}}+\frac{m_{15}}{\zeta^{15}}+\frac{m_{19}}{\zeta^{19}}\right)}\right)\\+\left(\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\phi_0'(\zeta)\right)\end{array}\right] \quad (12)$$

And so from equation 5

$$\phi(\zeta)=\frac{R}{2}\left[\begin{array}{l}(B+p+iT)\left(\zeta+\frac{m_3}{\zeta^3}+\frac{m_7}{\zeta^7}+\frac{m_{11}}{\zeta^{11}}+\frac{m_{15}}{\zeta^{15}}+\frac{m_{19}}{\zeta^{19}}\right)\\-\left(\begin{array}{l}2(B+p+iT)\left(\frac{m_3}{\zeta^3}+\frac{m_7}{\zeta^7}+\frac{m_{11}}{\zeta^{11}}+\frac{m_{15}}{\zeta^{15}}+\frac{m_{19}}{\zeta^{19}}\right)\\+\frac{(B'-iC')}{\zeta}\end{array}\right)\end{array}\right]$$

$$\psi(\zeta)=\frac{R}{2}\left[\begin{array}{l}(B'+iC')\left(\zeta+\frac{m_3}{\zeta^3}+\frac{m_7}{\zeta^7}+\frac{m_{11}}{\zeta^{11}}+\frac{m_{15}}{\zeta^{15}}+\frac{m_{19}}{\zeta^{19}}\right)\\-\left(\begin{array}{l}\left(\frac{2(B+p+iT)\left(\frac{1}{\zeta}\right)+B'+iC'}{\left(\frac{m_3}{\zeta^3}+\frac{m_7}{\zeta^7}+\frac{m_{11}}{\zeta^{11}}+\frac{m_{15}}{\zeta^{15}}+\frac{m_{19}}{\zeta^{19}}\right)}\right)\\+\left(\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\phi_0'(\zeta)\right)\end{array}\right)\end{array}\right] \quad (12)$$

From the stress functions obtained above(i.e. ψ(ζ);Φ(ζ)) the stresses in polar co-ordinate can be found as follows:

$$\sigma_x+\sigma_y=\sigma_\theta+\sigma_\rho=4\,\mathrm{Re}[\phi'(\zeta)/\omega'(\zeta)]$$

$$\sigma_\theta-\sigma_\rho+2i\tau_{\rho\theta}=\left[\begin{array}{l}\left(\sigma_y-\sigma_x+2i\tau_{xy}\right)\\\left(\frac{\zeta^2\omega'(\zeta)}{\rho^2\overline{\omega'(\zeta)}}\right)\end{array}\right] \quad (13)$$

VII Results and Discussion

The stress functions obtained above are the generalized solutions, using which stress field around square hole under different loading conditions is presented. The material parameters taken are: E=200GPa; G=80GPa and ν=0.3. The following cases have been considered.

1-Plate subjected to uniaxial tension at infinite distance.
2-Plate subjected to biaxial tension at infinite distance.
3-Plate subjected to shear at infinite distance.
4-Plate having hole subjected to internal pressure on boundary.
5-Plate having hole subjected to internal shear on the hole boundary

1) Plate subjected to uniaxial tension at infinite distance

The infinite plate with single hole is subjected to uniaxial tension (λ=0). Hole orientation angle α is taken zero. The stresses in X-direction, Y-direction, shear stress and Von-mises stresses are calculated at hole boundary using MATLAB6.5. The stress fields obtained from the present method are also compared with Finite element solution using ANSYS(Refer fig2 and fig 3).The table 1 gives comparison of the some of results.

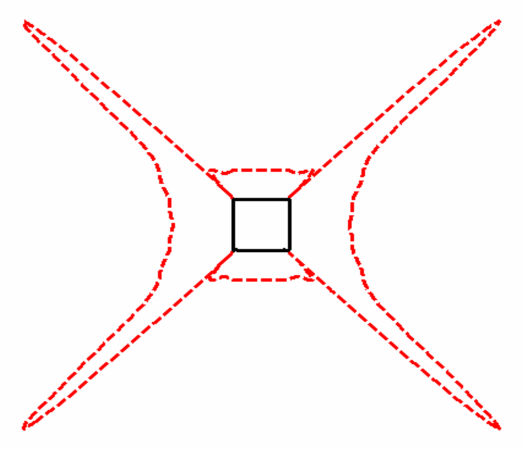

Fig 2 Von-Mises stress (Present method) (λ=0)

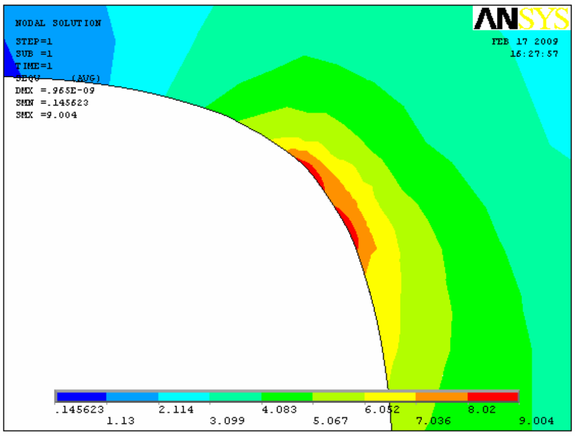

Fig 3 Von-Mises stress (ANSYS) (λ=0)

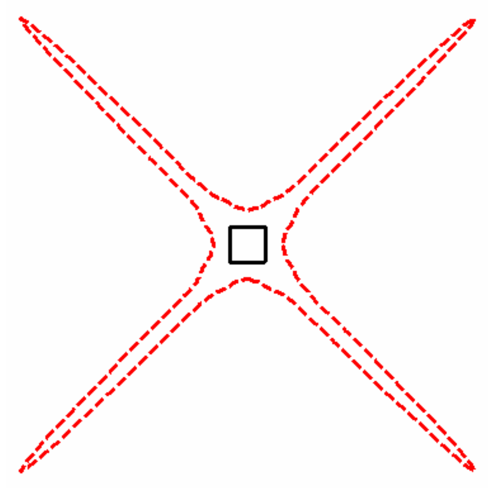

Fig.4 Von-Mises stress in plate (λ=1)(Present method)

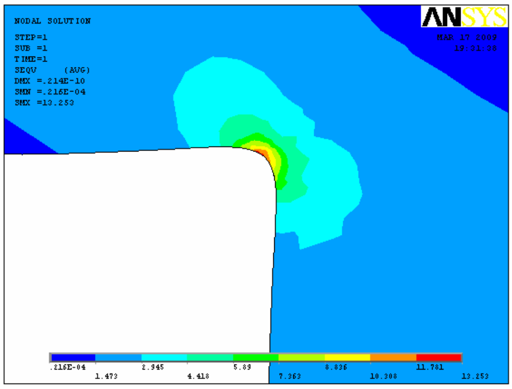

Fig.5 Von-Mises stress in plate (λ=1)(ANSYS)

TABLE I
COMPARISON OF RESULTS OBTAINED BY PRESENT METHOD AND ANSYS (SUBJECTED TO UNIAXIAL AND BIAXIAL TENSION)

		Uniaxial Load(λ=0)		
		Present method	ANSYS	Percentage difference
Stress in X-direction	Max	3.613	3.632	0.523128
Stress in Y-direction	Max	7.731	7.772	0.527535
Shear stress	Max	3.387	3.436	1.426077
Von-mises stress	Max	8.736	9.004	2.976455

2) Plate subjected to biaxial tension at infinite distance

Here stress distribution around same hole shape under equi-biaxial loading (λ=1) is shown in fig.4 while fig.5 shows same stress distribution using FEM (ANSYS) The comparison of some of the results with that of ANSYS is tabulated below (Table 2).

TABLE II
COMPARISON OF RESULTS OBTAINED BY PRESENT METHOD AND ANSYS (SUBJECTED TO SHEAR LOAD AT INFINITY)

		Biaxial Load(λ=1)		
		Present method	ANSYS	Percentage difference
Stress in X-direction	Max	9.956	9.951	-0.05025
Stress in Y-direction	Max	9.956	9.965	0.090316
Shear stress	Max	7.127	7.31	2.50342
Von-mises stress	Max	14.25	14.682	2.942378

3) Plate subjected to shear at infinite distance

In this case Von-Mises stress patterns for infinite plate with hole rotated by 45^0 under shear loading at infinity. The comparison of results derived from present method and FEM (ANSYS) can be seen from fig 7 and fig 8.Also, some of the extremum results are compared separately (refer table 3).

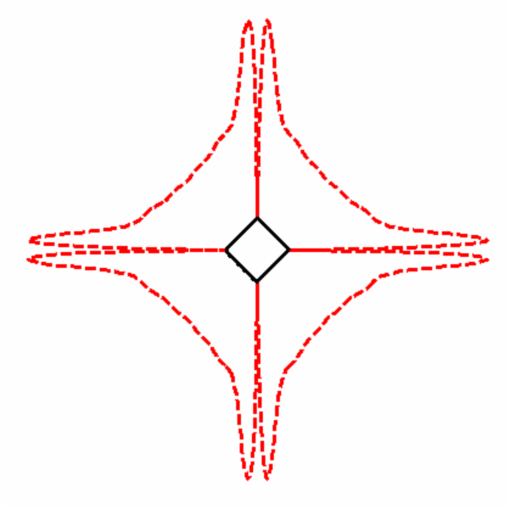

Fig.6 Von-Mises stress in plate (λ=-1)(Present method)

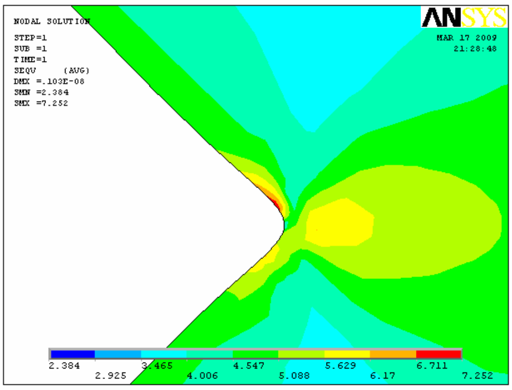

Fig. 7 Von-Mises stress in plate (λ=-1)(ANSYS)

TABLE 3

COMPARISON OF RESULTS OBTAINED BY PRESENT METHOD AND ANSYS (SUBJECTED TO SHEAR LOAD AT INFINITY)

		Shear Load(λ=-1)(α=45⁰)		
		Present method	ANSYS	Percentage difference
Stress in X-direction	Max	5.239	5.31	1.3371
Stress in Y-direction	Max	5.239	5.224	-0.28714
Shear stress	Max	-3.565	-3.55	-0.42254
Von-mises stress	Max	7.339	7.252	-1.19967

4) Plate with hole subjected to internal Pressure/Shear

Solution is also capable to handle pressure or shear on hole boundary. Fig.9 and Fig.10 shows comparison of Von-Mises stress for internal pressure (P=1). Comparison of results for this loading condition is shown in table 4.

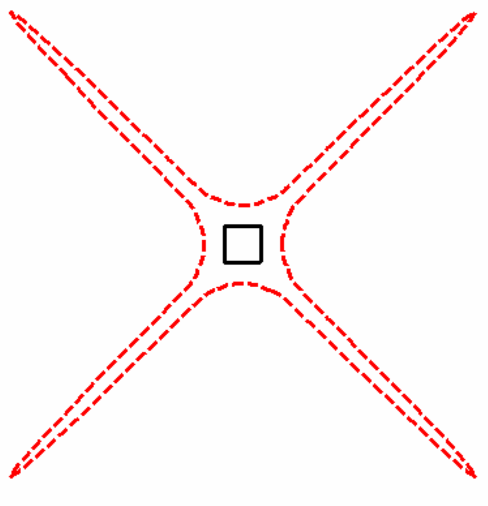

Fig.8 Von-Mises stress in plate (p=1) (Present method)

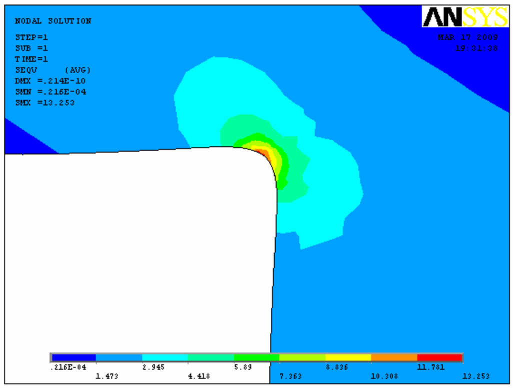

Fig.9 Von-Mises stress in plate(λ=-1)(Present method)

TABLE IV

COMPARISON OF RESULTS OBTAINED BY PRESENT METHOD AND ANSYS (SUBJECTED TO INTERNAL PRESSURE)

		Shear Load(p=-1)		
		Present method	ANSYS	Percentage difference
Stress in X-direction	Max	8.956	8.948	-0.08941
Stress in Y-direction	Max	8.956	8.879	-0.86721
Shear stress	Max	7.127	7.115	-0.16866
Von-mises stress	Max	13.78	13.253	-3.97646

VIII CONCLUSIONS

The general solution presented in this paper is very much useful to study stress field around square hole under different type of in-plane loading. A detailed parametric study can be easily made using the present solution by merely introducing the mapping function, the orientation angle and the biaxial loading factor. The computer implementation is easier and faster then finite element method and the results are quite comparable. With slight modification in mapping function same formulation can be used to find stress fields around different hole shapes and also for hole with one or two cracks emanating from it, and hole with cusps.

X REFERANCES

[1] Muskhelishvili, N.I., *Some basic problems of the mathematical theory of elasticity*, P. Noordhooff Ltd., Groningen, The Netherlands, 1963.

[2] Gao X L , "A general solution of an infinite elastic plate with an Elliptic hole under biaxial loading", *International Journal of Pressure Vessels and Piping*,67,1996,pp.95-104.

[3] V G Ukadgaonker , D K N Rao "A general solution for stresses around holes in symmetric laminates under in plane loading", *Journal of Composite Structures 49,2000, pp.339-354*