THE ATMOSPHERIC PRESSURE IN DRY REGION AND VELOCITY OF INFILTERED WATER IN GROUND WATER INFILTERATION PHENOMENON

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ABSTRACT

The non-linear partial differential equation of groundwater infiltration has been solved with appropriate boundary conditions and height of water mound is obtained in form of error function.

The atmospheric pressure in air of the dry region have been obtained by using height of water mound and then the velocity of infiltered water have been obtained by using of Darcy's law. The graphical presentation is given by using MAT LAB coding.

Keywords: Infiltration, Pressure, velocity, Darcy's law

1 INTRODUCTION

When fluid infiltered in porous medium (unsaturated soil), its velocity decreases as soil becomes saturated such phenomenon is called infiltration. It has been discussed by different authors from different viewpoints; for example, Darcy(1856), A.E.Scheidegier(1960)], M.Muskat(1946) and Jacob Bear(1946). It has also been discussed in homogeneous porous media as well as heterogeneous porous media by Verma(1967), Mehta & Verma(1977), Mehta & Patel(2007) and Mehta & Desai(2010).

The infiltration phenomenon is useful to control salinity of water; contamination of water and agriculture purpose and it is also useful in chemical engineering, nuclear waste disposable problems.

Such problems are also useful to measure moisture content of water in vertical one dimensional ground water recharge and dispersion of any fluid in porous media. It has been discussed by M.N.Mehta(2006), Mehta & Patel(2007), Mehta & Yadav(2007), Mehta & Joshi(2009), Mehta & Mehar(2010) from different viewpoints.

2 STATEMENT OF THE PROBLEM

Infiltration is the process by which water on the ground surface enters the unsaturated soil. The purpose of this paper is to present physically meaningful technique to determine effective height of the water table as a measure of initial storage capacity of a basin. Thus first an equation is derived for mean water table height on the basis of hydraulic theory of ground water by means of Boussinesq's equation; second is to solve that equation by using perturbation method. Then with the help of height of free surface, the atmospheric pressure in dry region and velocity of infiltered ground water is to calculate by using Darcy's law.

3 MATHEMATICAL FORMULATION

It is assumed that:

(i) The stratum has height h_m and lies on top of a horizontal impervious bed, which is labeled as z = 0;

(ii) Ignore the transversal variable y; and

(iii) The water mass which infiltrates the soil occupies a region described as $\Omega = \{(x, z) \in R: z \leq h(x, t)\}$.

Clearly, $0 \leq h(x, t) \leq h_m$, h_m maximum height of free surface and the free boundary function h is also an unknown. For the sake of simplicity and for the practical computation after introducing a suitable assumption, the hypothesis of almost horizontal flow, i.e., we assume that the flow has an almost horizontal speed.

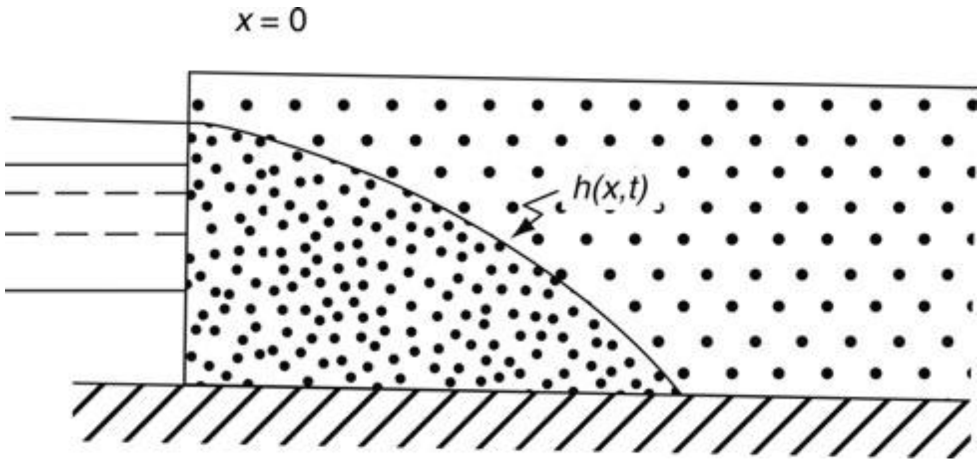

Figure 3.1: A schema of ground infiltration

$u \approx (u, 0)$, so that h has small gradients. It follows that in the vertical component of the momentum equations

$$\rho\left(\frac{du_z}{dt} + u \cdot \nabla u_z\right) = -\frac{\partial p}{\partial z} - \rho g$$

Neglecting the inertial term (the left-hand side), Integration in z gives for this first approximation $p+\rho gz=$ constant.

Now calculate the constant on the free surface z = h(x, t).

If we impose continuity of the pressure across the interface, we have p = 0 (assuming constant atmospheric pressure in the air that fills the pores of the dry region z > h(x, t)).

$\Rightarrow$ p = ρg (h − z). (1)

In other words, the pressure is determined by means of the hydrostatic approximation.
Now using mass conservation law, taking a section S = (x, x + a) × (0, C),

$$\phi \frac{\partial}{\partial t}\int_{x}^{x+a}\int_{0}^{h} dz dx = -\int_{\partial s} u\cdot n\, dl \tag{2}$$

where ϕ is the porosity of the medium, i.e., the fraction of volume available for the flow circulation, and u is the velocity, which obeys Darcy's law in the form that includes gravity effects

$$u=-\frac{k}{\mu}\nabla(p+\rho gz) \tag{3}$$

On the right-hand lateral surface we have u · n ≈ (u, 0) · (1, 0) = u, i.e., − (k/μ)px, while on the left-hand side we have −u.

Using the formula for p and differentiating in x, we get

$$\phi\frac{\partial h}{\partial t}=\frac{\rho gk}{\mu}\frac{\partial}{\partial x}\int_{0}^{h}\frac{\partial}{\partial x}h\,dz \tag{4}$$

We thus obtain Boussinesq's equation.

$$\frac{\partial h}{\partial t}=\beta\frac{\partial^2}{\partial x^2}\left(h^2\right) \tag{5}$$

where constant $\beta=\rho gk/2\varepsilon\mu$.

4 SOLUTION OF THE PROBLEM

Choose dimension less variable X = $\frac{x}{1}$ and T = β t, then equation (5) can be written as

$$\frac{\partial h}{\partial T} = \frac{\partial^2}{\partial X^2}\left(h^2\right) \tag{6}$$

With the boundary conditions

h(0, T) = h_m when x = 0 , any T > 0 (7)

h(X, 0) = h_0 when T = 0 , any X (8)

It is a fundamental equation in groundwater infiltration. Once h(x, t) is calculated, we may calculate the pressure via (1) and then the speed by means of Darcy's law.

Using the transformation $\frac{h}{h_0} = F(\theta)$ *where* $\theta = \frac{2T}{X^2}$, (9)

Combining equation (6) and (9),

$$2\theta\left[2\theta\left(FF'' + F'^2\right) + 3FF'\right] = F' \tag{10}$$

Subtitling value of (10) implies that $\theta \to 0$ as $T > 0$ for all finite value of X. It is also implies that $\theta \to \infty$ as $X \to 0$ for all finite value of T. Hence by using (10) the boundary conditions,

$h(0, \theta) = h_m$ at X = 0, $\theta > 0$ gives F $(\infty) = \frac{h_m}{h_0} = M > 1$ (11)

And

$h(X, 0) = h_0$ at $\theta = 0$ for finite value of X gives F(0) = 1 (12)

To write conditions (7) and (8) in more convenient for, we introduce new function $f(\theta)$ defines by

$F(\theta) = 1 + (M-1)\, f(\theta)$; $(M \neq 1)$ (13)

So that we can obtained

$f(0) = 0$ and $f(\infty) = 1$ (14)

Now equation (10) reduce to

$$2\theta\left[2\theta\left\{(1 + (M-1)f)f'' + (M-1)f'^2\right\} + 3\,(1 + (m-1)f)f'\right] = f' \tag{15}$$

Equation (15) is a nonlinear differential equation which can be solved for a given set of initial and boundary conditions; if height h_0 is slightly different from h_m we may choose

$\varepsilon = (M - 1)$, where $0 < \varepsilon < 1$.

Putting $\varepsilon = (M - 1)$ in (15), we get

$$2\theta \left[2\theta \left\{(1+\varepsilon f)f'' + \varepsilon f'^2\right\} + 3(1+\varepsilon f)f''\right] = f' \tag{16}$$

Now to find the solution of equation (16), using perturbation technique, let the solution be expressed in the form,

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots\dots\dots\dots \tag{17}$$

Where f_0, f_1, f_2 .... are continuous and differentiable functions of θ to be determined and $0 < \varepsilon < 1$ is the perturbation parameter. Subtitling (17) in equation (16) and then equating coefficient of ε^r to zero, for r=0, 1,2 .... we obtain,

$$2\theta \left[2\theta f'_0 + 3f''_0\right] = f'_0 \tag{18}$$

$$2\theta \left[2\theta \left(f_0 f''_0 + {f'_0}^2 + f_1\right) + 3f_0 f'_0 + 3f'_1\right] = f'_1 \tag{19}$$

and so on.

Now the equation (18) may be written as

$$\frac{f''_0}{f'_0} = \frac{(1-6\theta)}{4\theta^2} \tag{20}$$

Its solution can be given as,

$$f_0 = c\int_0^\theta \theta^{-3/2} . e^{-1/4\theta}\, d\theta + c_1 \tag{21}$$

Where c and c_1 are constant of integration and this value can be obtained with help of conditions (14). Using the properties of error function, we obtained,

$$c = \frac{1}{2\sqrt{\pi}} \text{ and hence from (21) is } f_0 = c\int_0^\theta \theta^{-3/2} . e^{-1/4\theta}\, d\theta \tag{22}$$

Hence the required solution of (18) is given by

$$f_0(\theta) = \operatorname{erfc}\left(\frac{1}{2\sqrt{\theta}}\right) \tag{23}$$

Since, $f_0(\theta) = \operatorname{erfc}\left(\frac{1}{2\sqrt{\theta}}\right) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2\sqrt{\theta}}} e^{-h^2}\, dh$

Differentiating with respect to θ gives,

$$f_0'(\theta) = \frac{1}{2\sqrt{\pi}}\,\theta^{-3/2}\,e^{-1/4\theta} \quad \text{and} \quad f_0''(\theta) = \frac{1}{8\sqrt{\pi}}\,(1-6\theta)\,\theta^{-7/2}\,e^{-1/4\theta} \tag{24}$$

Now substituting (24), in the equation (19) reduce to

$$4\theta^2 f_1'' + (6\theta-1)\,f_1' = -\left[\left(\frac{1}{\pi}\right)\theta^{-1}e^{-1/2\theta} + \left(\frac{1}{2\sqrt{\pi}}\right)\theta^{-3/2}\,e^{-1/4\theta}\,\text{erfc}\left(\frac{1}{2\sqrt{\theta}}\right)\right] \tag{25}$$

Thus is linear differential equation in f_1', with constant coefficient and its solution may be obtained in usual manner. Thus we obtained

$$f_1(\theta) = -\frac{1}{8\pi}\left[4\pi\left[\text{erfc}\left(\frac{1}{2\sqrt{\theta}}\right)\right]^2 - 4\sqrt{\pi}\left(\theta^{-1/2}\,\theta^{-1/4\theta}\right)\text{erfc}\left(\frac{1}{2\sqrt{\theta}}\right) + 8e^{-1/2\theta}\right] + \sqrt{\pi}c_1\left[\text{erfc}\left(\frac{1}{2\sqrt{\theta}}\right)\right] + c_2 \tag{26}$$

Where c_1 and c_2 are constants of integration.

Substituting values of f_0 and f_1 from (23) and (26) in (17) and retaining only terms in **ε** only, we get

$$f(\theta) = \text{erf}\left(\frac{1}{2\sqrt{\pi}}\right) + \varepsilon\left[\frac{1}{2}\left[\text{erfc}\left(\frac{1}{2\sqrt{\pi}}\right)\right]^2 + \frac{1}{2\sqrt{\pi}}\,\theta^{-1/2}\,e^{-1/4\theta}\,\text{erfc}\left(\frac{1}{2\sqrt{\theta}}\right) - \frac{1}{\pi}\,e^{-1/2\theta} + 2\pi\ \text{c1erfc12}\theta\text{+c2} \right. \tag{27}$$

The values of c_1 and c_2 are obtained from the condition in (14) with help of properties of error function. Thus the solution (26) can be expressed as

$$f(\theta) = \text{erfc}\left(\frac{1}{2\sqrt{\theta}}\right) + (\varepsilon-1)\left[\left(\frac{1}{2\sqrt{\theta}}\right) - 0.5serfc\left(\frac{1}{2\sqrt{\theta}}\right) + 0.282\left(e^{-1/2}e^{-1/4\theta}\right) + 0.8183 - 0.3183e{-}12\theta\right. \tag{28}$$

Since, $F(\theta) = 1 + (M-1)\,f(\theta)$

$$F(\theta) = 1 + \varepsilon\left\{erfc\left(\frac{1}{2\sqrt{\theta}}\right) + (\varepsilon-1)\left[erf\left(\frac{1}{2\sqrt{\theta}}\right) - 0.5serfc\left(\frac{1}{2\sqrt{\theta}}\right) + 0.282\left(e^{-1/2}e^{-1/4\theta}\right) + 0.8183{-}0.3183e{-}12\theta\right.\right. \tag{29}$$

Replacing θ *by* $\dfrac{2T}{X^2}$ *and by* (9),

$$h(X,T) = h_0\left[1 + \varepsilon\left\{\left(erfc\left(\frac{X}{2\sqrt{2T}}\right) + (\varepsilon-1)\left[erf\left(\frac{X}{2\sqrt{2T}}\right) - 0.5serfc\left(\frac{X}{2\sqrt{2T}}\right) + 0.282e{-}12e{-}X28T{+}0.8183{-}0.3183e{-}X24T\right.\right.\right.\right. \tag{30}$$

Changing into original variables,

$$h(x,t) = h_0\left[1+ \varepsilon\left\{\left(erfc\left(\frac{x}{2\sqrt{2\beta t}}\right) + (\varepsilon-1)\left[erf\left(\frac{x}{2\sqrt{2\beta t}}\right) - 0.5serfc\left(\frac{x}{2\sqrt{2\beta t}}\right) + \right.\right.\right.\right.$$
0.282e−12e−x28βt+0.8183−0.3183e−x24βt (31)

This provides the approximate solution for $0 < \varepsilon < 1$ which gives the height of free surface of water mound.

The atmospheric pressure in the air that fills the pour of dry region $z > h(x,t)$ is

$$P = \rho g\left[h_0\left[1+\right.\right.$$
εerfc
x22βt+ε−1erfx22βt−0.5serfcx22βt+0.282e−12e−x28βt+0.8183−0.3183e−x24βt− z
(32)

$$p+\rho gz = \rho g h_0\left[1+ \varepsilon\left\{\left(\mathrm{erfc}\left(\frac{x}{2\sqrt{2\beta t}}\right) + (\varepsilon-1)\left[\mathrm{erf}\left(\frac{x}{2\sqrt{2\beta t}}\right) - 0.5\mathrm{serfc}\left(\frac{x}{2\sqrt{2\beta t}}\right) + \right.\right.\right.\right.$$
0.282e−12e−x28βt+0.8183−0.3183e−x24βt (33)

which represent the pressure determined by mean of hydrostatic approximation.

Also u is the velocity of infiletered water which obeys Darcy's law in the form that includes Gravity effect is

$$u = -\frac{k}{\mu}\nabla(p+\rho gz)$$

Using value of pressure from (11) we get velocity of infiltered water,

$$u = -\frac{k}{\mu}\left[\nabla\left\{\rho g h_0\left[1+ \varepsilon\left\{\left(\mathrm{erfc}\left(\frac{x}{2\sqrt{2\beta t}}\right) + (\varepsilon-1)\left[\mathrm{erf}\left(\frac{x}{2\sqrt{2\beta t}}\right) - 0.5\mathrm{serfc}\left(\frac{x}{2\sqrt{2\beta t}}\right) + \right.\right.\right.\right.\right.\right.$$
0.282e−12e−x28βt+0.8183−0.3183e−x24βt (34)

$$u = -\frac{k\rho g h_0 \varepsilon^2}{\mu\sqrt{2\pi\beta t}}\; e^{-x^2/8\beta t}\left[1-\frac{\varepsilon-1}{\varepsilon}\left\{0.5+\frac{0.141}{\sqrt{2\beta te}}-\frac{0.3183\ e^2}{\sqrt{2\beta t}}\right\}\right] \qquad (35)$$

5 NUMERICAL & GRAPHICAL PRESENTATIONS

Numerical and graphical presentations of equations (31), (32) and (35) have been obtained by using MATLAB coding. Figure (5.1) shows the graph of height h vs. X for time T = 0.1, 0.2, 0.3, 0.4, 0.5, figure (5.2) shows pressure P vs. X for time T= 0.1, 0.2, 0.3, 0.4, 0.5 and figure (5.3) shows the graph of velocity u vs. X for time T= 0.1, 0.2, 0.3, 0.4, 0.5.

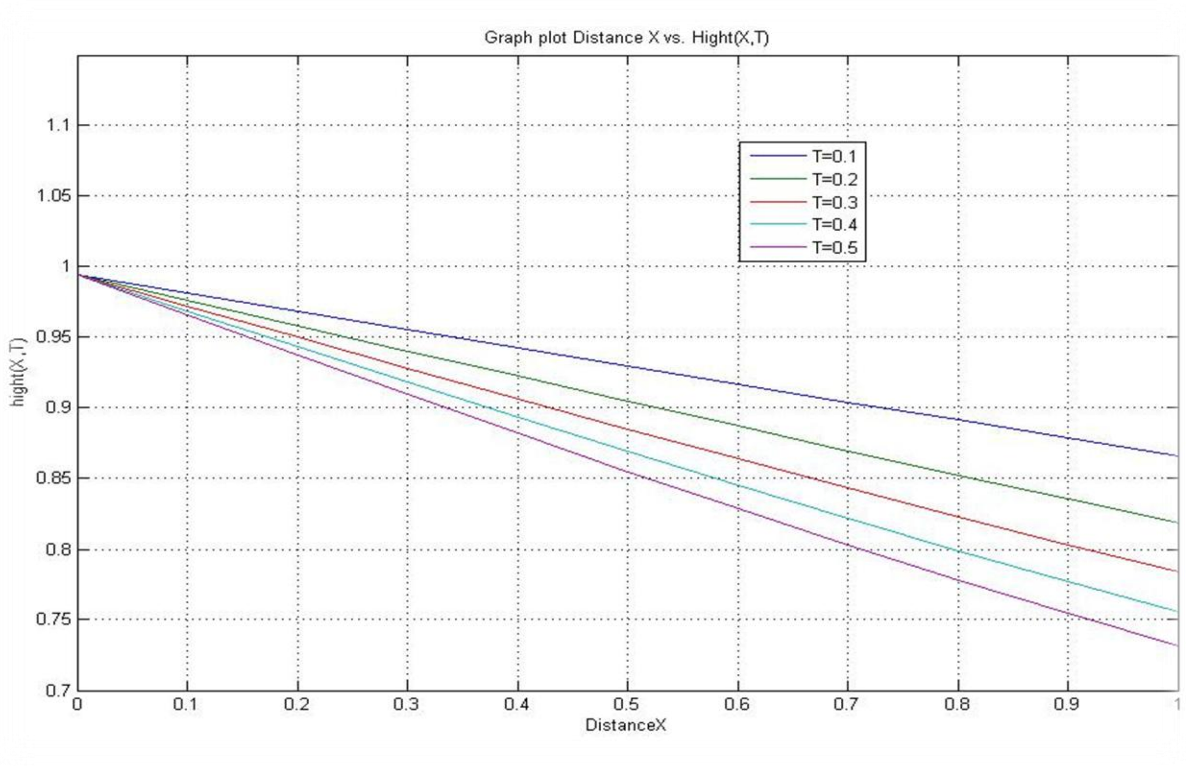

Figure 5.1

Table 1: ho = 0.5, **ε** = 0.99, g = 9.8, ρ = 0.1 are Fixed.

	$h(X,T)$				
X	T=0.1	T=0.2	T=0.3	T=0.4	T=0.5
0.0	0.9942	0.9942	0.9942	0.9942	0.9942
0.1	0.9814	0.9762	0.9722	0.9688	0.9658
0.2	0.9686	0.9582	0.9503	0.9435	0.9376
0.3	0.9558	0.9403	0.9284	0.9184	0.9097
0.4	0.9429	0.9224	0.9068	0.8936	0.8821
0.5	0.9300	0.9047	0.8853	0.8692	0.8551
0.6	0.9171	0.8871	0.8642	0.8452	0.8287
0.7	0.9043	0.8697	0.8434	0.8218	0.8031
0.8	0.8915	0.8524	0.8231	0.7989	0.7783
0.9	0.8787	0.8355	0.8031	0.7768	0.7545
1	0.8659	0.8188	0.7838	0.7555	0.7318

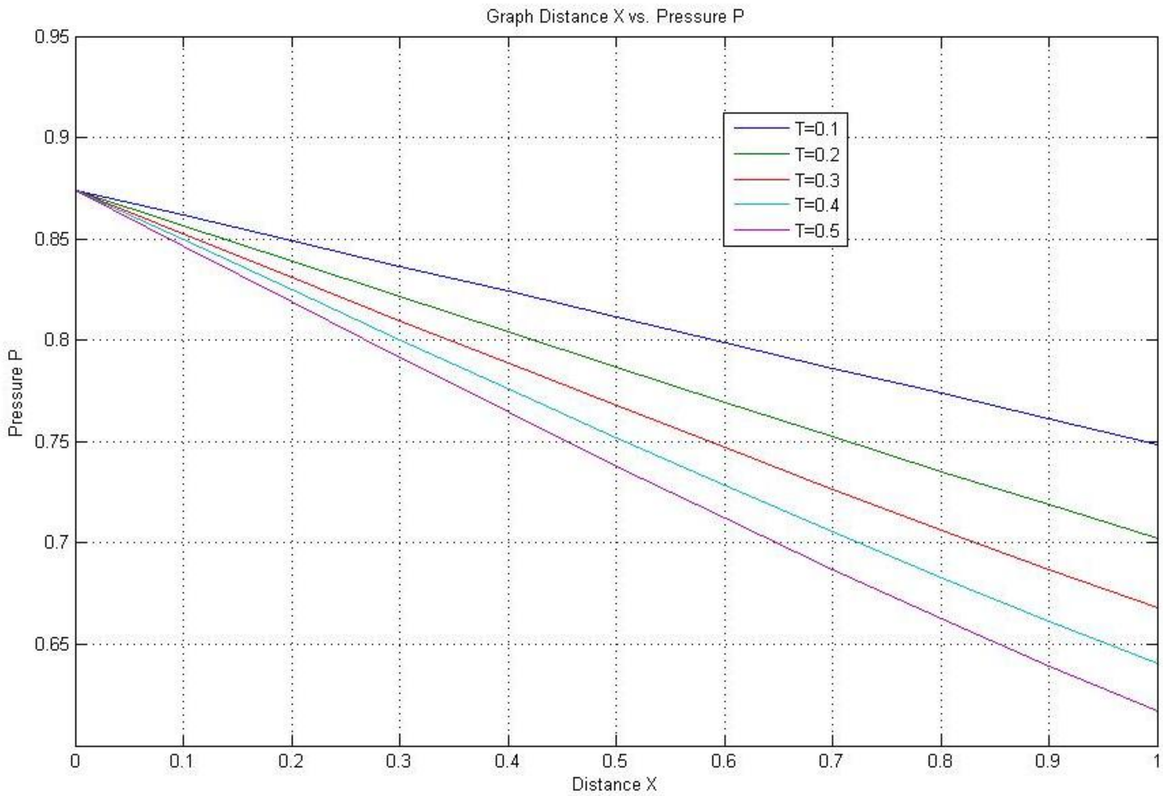

Figure 5.2

Table 2: ho = 0.5, **ε** = 0.99, g = 9.8, ρ = 0.1 are Fixed.

	P				
X	T=0.1	T=0.2	T=0.3	T=0.4	T=0.5
0.0	0.8743	0.8743	0.8743	0.8743	0.8743
0.1	0.8618	0.8567	0.8527	0.8494	0.8465
0.2	0.8493	0.8391	0.8313	0.8247	0.8189
0.3	0.8366	0.8215	0.8089	0.8001	0.7915
0.4	0.8240	0.8040	0.7896	0.7757	0.7645
0.5	0.8114	0.7866	0.7676	0.7757	0.7380
0.6	0.7988	0.7693	0.7469	0.7283	0.7120
0.7	0.7862	0.7523	0.7266	0.7053	0.6870
0.8	0.7737	0.7354	0.7066	0.6830	0.6628
0.9	0.7612	0.7188	0.6871	0.6613	0.6395
1	0.7486	0.7024	0.6681	0.6404	0.6179

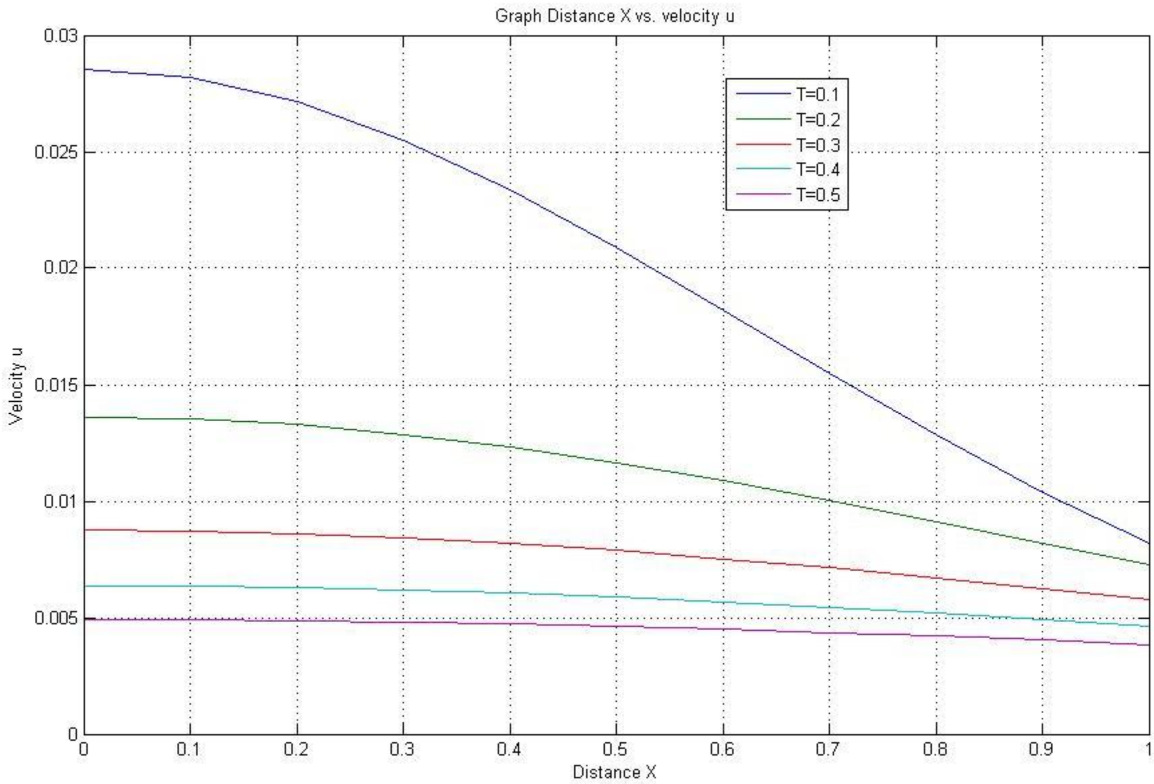

Figure 5.3

Table 3: ho = 0.5, **ε** = 0.99, g = 9.8, ρ = 0.1 are Fixed.

	U				
X	T=0.1	T=0.2	T=0.3	T=0.4	T=0.5
0.0	0.0285	0.0136	0.0087	0.0064	0.0049
0.1	0.0282	0.0135	0.0087	0.0063	0.0049
0.2	0.0271	0.0133	0.0086	0.0063	0.0049
0.3	0.0255	0.0129	0.0084	0.0062	0.0048
0.4	0.0234	0.0123	0.0083	0.0060	0.0047
0.5	0.0209	0.0116	0.0082	0.0059	0.0046
0.6	0.0182	0.0109	0.0079	0.0057	0.0045
0.7	0.0155	0.0100	0.0075	0.0054	0.0044
0.8	0.0128	0.0091	0.0071	0.0052	0.0042
0.9	0.0104	0.0082	0.0067	0.0049	0.0040
1	0.0082	0.0073	0.0062	0.0046	0.0038

6 COMPARISON WITH THE POWER SERIES SOLUTION

Reference: Paper titled "Power Series Solution of Boussinesq's equation arising in ground water infiltration phenomena" by K. R. Patel and M. N. Mehta communicated for the publication at GAMS.

(a) Height at different distance when T = 0.1, 0.2, 0.3, 0.4, 0.5 AND ω = 0.1, ρ=0.1, z=0.1 and g=9.8 Fixed.

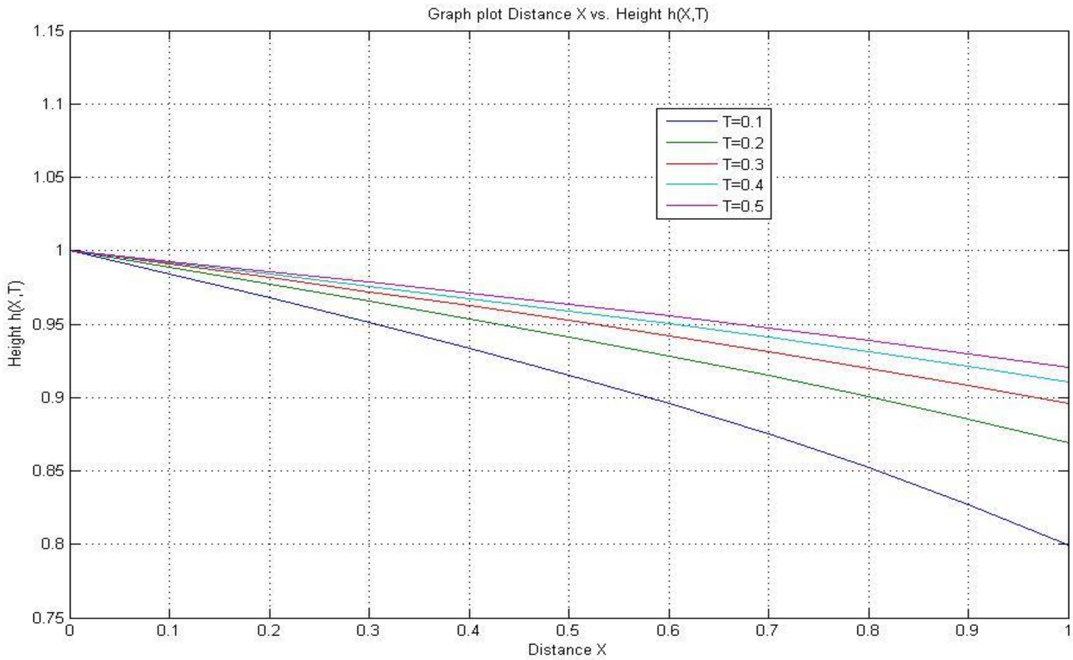

Figure 6.1

(b) Pressure at different distance when T = 0.1, 0.2, 0.3, 0.4, 0.5 AND ω = 0.1, ρ=0.1, z=0.1 and g=9.8 Fixed.

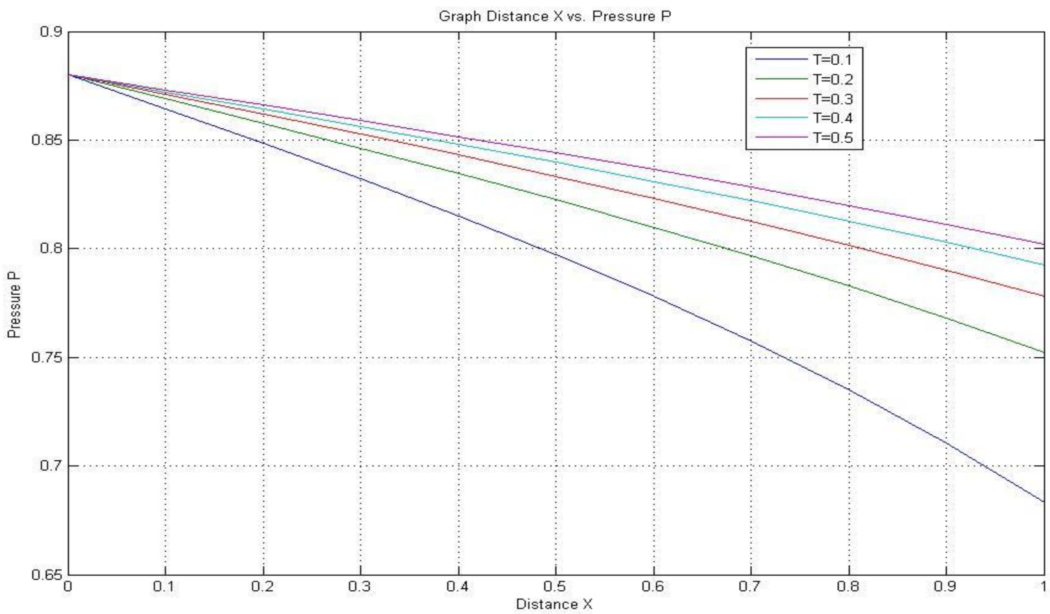

Figure 6.2

(c) Velocity at different distance when T = 0.1, 0.2, 0.3, 0.4, 0.5 AND ω = 0.1, ρ=0.1, z=0.1 and g=9.8 Fixed.

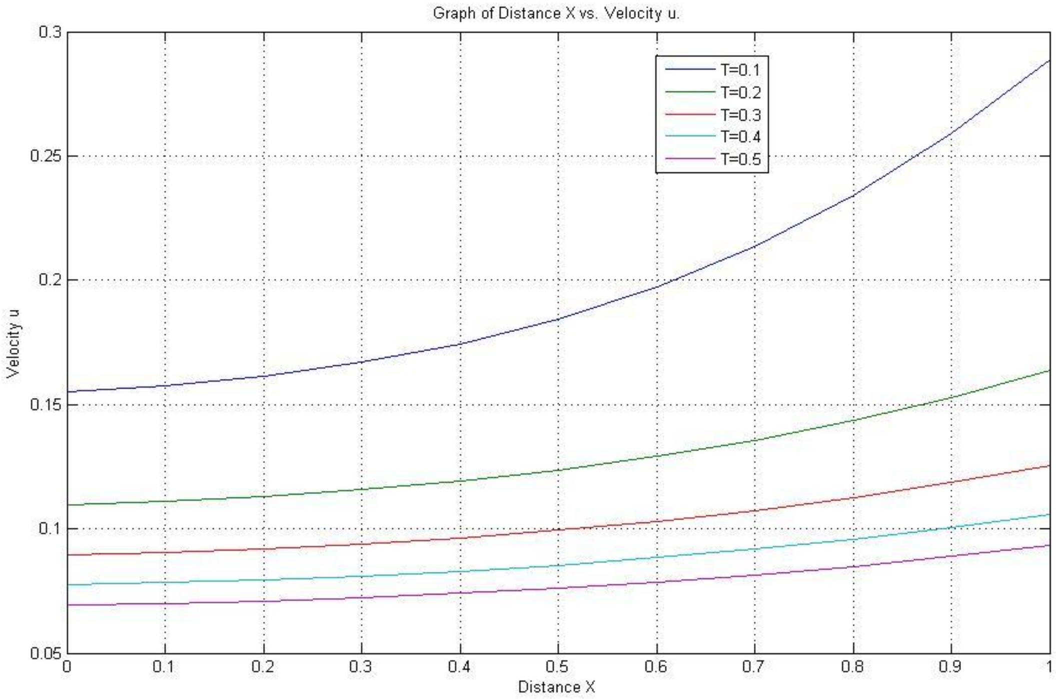

Figure 6.3

Comparing figure 5.1 with figure 6.1, figure 5.2 with 6.2 and figure 5.3 with 6.3, it can be observe that the solution obtained by perturbation technique is matching with the power series solution and consistent with physical nature.

7 CONCLUDING REMARK

Equation (31) shows the height of water mound in infiltration phenomena at any x for t > 0.

Equation (32) is the atmospheric pressure in the air of the dry region and the speed of unfiltered water can be measured by equation (35).

We can conclude from equation (31) that the height of the water mound is expressed in terms of error function and complimentary error function. Also when x increases, the height h(x,t) decreases for any time t > 0 which is consistent with physical phenomena.

We can conclude from equation (32) is that the atmospheric pressure in dry region is expressed in terms of error function and complimentary error function which is negative exponential function. Hence when x increases, atmospheric pressure (P) decreases in dry region for any value of z.

Also we can conclude from equation (35) that the velocity of infiltered water is expressed in terms of negative exponential term for any time t > 0 , for $0 < \varepsilon < 1$. Hence velocity of infiltered water (u) is also decreasing when distance travelled x increases.

The graph of equation (30) shows h verses x for any time t is given by using MAT LAB that when x increases , the height h(x,t) decreases for any time t > 0 which is consistent with physical phenomena as shown by figure 5.1.

The graph of equation (32), P verses x is given by using MAT LAB shows that when x increases, the atmospheric pressure P decreases for any time t > 0 which is shown by figure 5.2.

The graph of equation (35), u verses x is given by using MAT LAB shows that when x increases, the velocity of infiltered water is also decreases which are the validity of Darcy's law in Porous media which is shown by figure 5.3.

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