"Vibration and Fatigue Analysis of Composite Materials"

A Major Project Report

Submitted in Partial Fulfillment of the Requirements for
The Degree of
MASTER OF TECHNOLOGY
IN
Mechanical Engineering

(CAD/CAM)

By

Saurabh Shrivastava

(06MME013)

Guided By:

Prof. S.J. Joshi

Department of Mechanical Engineering
INSTITUTE OF TECHNOLOGY
NIRMA UNIVERSITY OF SCIENCE & TECHNOLOGY,
AHMEDABAD 382 481

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CERTIFICATE

This is to certify that the Major Project Report entitled

"Vibration and Fatigue Analysis of Composite Materials"

submitted by **Saurabh Shrivastava (06MME013)** towards the partial fulfillment of the requirements for Master of Technology (Mechanical) in the field of CAD/CAM of Nirma University of Science and Technology is the record of work carried out by him under our supervision and guidance. The work submitted has in our opinion reached a level required for being accepted for examination. The results embodied in this major project work (Part 1) to the best of our knowledge have not been submitted to any other University or Institution for award of any degree or diploma.

Project Guides:

1/5/08

Prof. S.J. Joshi,
Assistant Professor,
Department of Mechanical Engg.
Institute of Technology,
Nirma University of Science
and Technology, Ahmedabad

Prof. V.R. Iyer,
Professor
Head of the Department of Mechanical Engg
Institute of Technology,
Nirma University of Science and
Technology,Ahmedabad

Prof. A.B. Patel,
Director,
Institute of Technology,
Nirma University of Science
and Technology, Ahmedabad.

CERTIFICATE

This is to certify that the major project titled

"Vibration and Fatigue Analysis Of Composite Materials"

Submitted by **Mr. Saurabh Shrivastava (06MME013)** towards the partial fulfillment of the requirements for Master of Technology (Mechanical) in the field of CAD/CAM of Nirma University of Science and Technology is found to be satisfactory and is approved.

Examiners:

1) 17/5/08

2)

3)

4)

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Dated: / /

Saurabh Shrivastava

06MME013

M.TECH. (CAD/CAM)

Abstract

Along with the advance of science and technology, composite materials including laminated composite plates have been widely used in various engineering field such as in aeronautic, astronautic ,auto- industries, submarine engineering ,nuclear technology ,and also for the fine construction such as circuit boards in electronic packages, thereby creating considerable interest in their analysis. It is inevitable to determine properties of lamina from known properties of fiber and matrix, natural frequencies, no. of cycles to failure as a part to ensure the safe design of the component.

As an alternative for determination of properties of lamina by experimental method which is expensive, method of cells is widely used. In the present work an algorithm is developed to determine properties of lamina from known properties of fiber and matrix.

An algorithm is also developed for determinations of natural frequency of plates made of composite materials with various boundary conditions. Results so obtained are compared by performing analysis using ANSYS.

Fatigue failure of composite material has always been an important problem in the application of composite structures. The fatigue behavior of composite materials is conventionally characterized by S - N curve or damage mechanisms. For every new material with a new lay-up, altered constituents or different processing procedure, a whole new set fatigue life tests has to be repeated. An algorithm is developed that includes the effect of stress ratio(R) and load frequency (f) for the prediction of fatigue life and residual strength of composite structures.

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Nomenclature

$w_i^{(\beta\gamma)}$	Displacement component of the centre of the subcell
$\bar{x}_2^{(\beta)}, \bar{x}_3^{(\gamma)}$	Displacements in the local coordinates
$\bar{\sigma}_{ij}$	Average stresses in the composite
$\bar{\sigma}_{ij}^{(\beta\gamma)}$	Average stresses in the subcells
εij	Average strains in the composite
εij by	Average strains in the subcells
b_{ij}	Effective elastic constants of the composite
Γ	Coefficient of thermal expansions
α_A	Axial coefficient of thermal expansion
α_T	Transverse coefficient of thermal expansion
Q_i	Average heat flux in the composite
E_A	Axial Young's modulus
E_T	Transverse Young's modulus
ν_A	Transverse Poisson'sratio
ν_T	Transverse Poisson's ratio
G_A	Axial shear modulus
f	Frequency(Hz)

N	Fatigue life (cycles) of the material
n	Number of fatigue cycles
R	Stress ratio (minimum applied stress/maximum applied stress)
t	Time
σ_{max}	Maximum applied stress in loading direction
σ_u	Ultimate stress of the virgin material in the loading direction
Xt	Ultimate Longitudinal stress of lamina
Yt	Ultimate Transverse stress of lamina
S	Ultimate Shear stress of lamina
U	Strain energy
T	Total kinetic energy
u, v, w	Displacement in plate in x,y,z directions

CHAPTER 1
INTRODUCTION

1.0 Introduction

Along with the advance of science and technology, composite materials including laminated composite plates have been widely used in various engineering field such as in aeronautic, astronautic ,auto- industries, submarine engineering ,nuclear technology ,and also for the fine construction such as circuit boards in electronic packages, thereby creating considerable interest in their analysis. In addition to their high strength / light weight, another important advantage of composite laminates is that structural properties can be tailored through changing the fiber angle and/or the number of plies .Various kinds of composite materials provides a wide range of selections for engineers. The need for more information on the behavior of laminated structural components, like plates, is clear. Rectangular plates are used in many engineering applications.

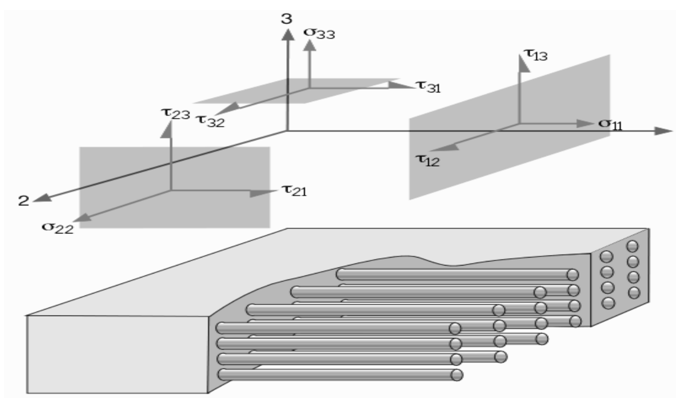


***Fig.*1** Cross-section Of Fibrous Composite

Vibration characteristics of composite structures are of considerable interest for their design and performance. The Rayleigh-Ritz method with algebraic polynomial displacement function is used to obtain natural frequencies of vibration for laminated composite plates. For rectangular plates there exist several combinations of boundary conditions among them cantilever rectangular plates (as used in turbo machinery, impeller and fan blades) , plates having all its edges simply supported (as used in structural mechanics, and other application), plates having all edges free (as used to detected damage of laminated panel by change in natural frequencies) are analyzed here. For example it is necessary to keep natural frequencies of structures well separated from their excitation frequencies in order to avoid resonance.

Classical example of resonance

The collapse of the Old Tacoma Narrows Bridge, nicknamed Galloping Gertie, in 1940 is sometimes characterized in physics textbooks as a classical example of resonance. This description is misleading, however. The catastrophic vibrations that destroyed the bridge were not due to simple mechanical resonance, but to a more complicated oscillation between the bridge and winds passing through it, known as aeroelastic flutter. Robert H. Scanlan, father of the field of bridge aerodynamics, wrote an article about this misunderstanding.

Fatigue failure of composite material has always been an important problem in the application of composite structures. Major structural components of an airplane are subjected to repeated loading (fatigue loading) during take-off and landing. It has been observed that for some composites the fatigue strength is substantially lower than the corresponding static strength. This repeated loading causes growth of fatigue cracks, which, if undetected, can lead to catastrophic failure.
Some examples of such accidents are:-

- In 1985, crash of a Japan airlines Boeing 747SR (short-range) jumbo-jet due to a catastrophic fatigue failure of rear pressure bulkhead which resulted in loss of 520 human lives.
- An accident involving an Aloha Airlines Boeing 737 jet in which much of upper half of a fuselage section was “blown off” during flight. The causes were fracture resulting from the undetected growth of multiple fatigue cracks.

The fatigue behavior of composite materials is conventionally characterized by S - N curve or damage mechanisms. For every new material with a new lay-up, altered constituents or different processing procedure, a whole new set fatigue life tests has to be repeated. Here an analytical method is presented that includes the effect of stress ratio and load frequency for the prediction of fatigue life and residual strength of composite structures.

1.1 Motivation of Work

Today composite is used in various fields, with high end applications as aircraft, wind turbines and so on. Very limited composite material‘s properties are available. The properties of lamina required are elastic properties, thermal properties, and strengths to find out laminate properties to carry out FEM and fatigue analysis. As all these properties are not readily available. To get all these properties expensive experimentation is required to be carried out. For rectangular plates cantilever rectangular plates (as used in turbo machinery, impeller and fan blades) , plates having all its edges simply supported (as used in structural mechanics, and other application), plates having all edges free (as used to detected damage of laminated panel by change in natural frequencies) is been used widely, so it is necessary to keep natural frequencies of structures well separated from their excitation frequencies in order to avoid resonance. Fatigue life of composite structures is the basic requirement as it will help to avoid sudden damage in composite structure which intimate the life after which the component is replaced

1.2 Objective of the Work

Objective of present work is to study the behavior of composite materials. The specific scope of the work is:

- Prediction of orthotropic / transversely isotropic lamina properties using micromechanics the very refined method “**Method of Cells**”.

- Obtain natural frequencies of vibration for laminated composite plates by Rayleigh-Ritz method with algebraic polynomial displacement function to avoid resonance.

- Prediction of fatigue behavior of composites using *S-N* curve approach and determination of residual strength and fatigue life at any no. of cycles (n) and any stress ratio (R)and frequency ratio(f).

CHAPTER 2
LITERATURE SURVEY

2.1 Structures And Properties

In general, composite materials are made by combining a matrix and reinforcement to create a material which has desirable characteristics and which are superior to either of the original materials individually. The majority of composite material usage is in a form of a fiber reinforced polymer matrix. E-poxy based systems are the most commonly used in matrix materials for polymer composites. E-poxy exhibits excellent properties overall, with excellent adhesion, high strength, low shrinkage, good corrosion protection and processing versatility. The second portion of a composite material is the reinforcement. Fiber reinforcement is used almost exclusively. The fiber material is the reinforcement in a composite material that gives the material the majority of its strength properties. The fibers generally have very high specific tensile strength and moduli, but depend on the matrix material to provide the transverse and compressive strength contributions. One of the types of fiber reinforcement such as carbon fibers offers light weight, high specific tensile modulus and strength, thus making them ideal for use in critical composite structures. They can be made from a variety of organic or petroleum fibers. Reinforcement can in the form of long fibers, short fibers, particles or whiskers. Composite materials have following advantages:

- High specific strength (strength to weight ratio) and specific modulus (stiffness to weight ratio).
- Much higher fatigue and endurance limit.
- Flexible applications-allows the design of materials and structures.

When comparing the properties of composites to monolithic materials, the stiffness or the strength of a composite may not be different, or perhaps lower than the metals. But when specific strength (strength to weight ratio) and specific stiffness (stiffness to weight ratio) are considered, composites generally outperform metals.

2.2 Properties of a Composite Based on Micromechanics

Some basic properties of composite materials can be estimated by using micromechanics. The properties of composites are related to the proportions of reinforcement and matrix. The proportions of matrix and reinforcement are either expressed by weight fraction (w) or by the volume fraction (v). In a composite the proportions of matrix and reinforcement is given by weight and volume fractions such as:

$$w_f + w_m = 1 \tag{2.1}$$

$$v_f + v_m = 1 \tag{2.2}$$

where subscripts f and m denote fiber and matrix.

Since the composite properties are dependent on proportions of matrix and reinforcement, we can compute the constitutive property of composite by the following equation:

$$X_C = X_f\, v_f + X_m\, v_m \tag{2.3}$$

where X represents an appropriate property of a composite. For example, the longitudinal modulus E11 of a composite can be computed as

$$E_{11} = E_f v_f + E_m v_m \tag{2.4}$$

Most properties of a composite are a complex function of a number of parameters as the constituents usually interact in a synergistic way, therefore the constitutive properties of the composite found by the law of mixtures are not fully accounted.

Composite laminates are designated in a manner indicating the number, type, orientation and stacking sequence of the plies. The configuration of the laminate indicating its ply composition is called a lay-up. The configuration indicates, in additions to the ply composition, the exact location or sequence of the various plies is called the stacking sequence. Table 2.1 Illustrates various types of laminate designations commonly used:

TABLE 2.1

Laminate Designation

Type	Sequence	Notation
Unidirectional ply	[90 / 90 / 90 / 90]	[90]4
Cross-ply symmetric	[0 / 90 / 90 / 0]	[90]s

where s = symmetric sequence

Number subscript = number of plies.

2.3 Natural Frequency of Vibration by Rayleigh-Ritz Principle

The Rayleigh-Ritz method is a fundamental technique which lends itself well to numerical methods. The variation approach finds application to solutions of problems of thermal transfer and diffusion, kinetics, magneto- and electrostatics, quantum theory, and so forth. The laws governing such phenomena are typically expressed through the Laplace, Poisson, diffusion, and eigenvalue equations. In this paper, attention will be focused on the eigenvalue problem.

Using the Rayleigh-Ritz principle, the eigenvalue problem may be rewritten as a matrix equation. In this form, solution of the problem becomes a matrix diagonalization procedure and many efficient means of solution become available. For an exact solution of the eigenvalue problem, treatment of an infinite dimensional matrix is required, and numerically this is awkward or effectively impossible. Thus, the problem is usually replaced by a truncated representation. A successive approximations approach proceeds by generating a trial solution, performing a matrix diagonalization for the truncated problem, improving the trial solution
and repeating the procedure. In this way, an improved trial solution is generated which approaches the solution of the full infinite dimensional problem [1].

$$\mathcal{H}_1 \rightarrow \mathcal{H}_2 \rightarrow \mathcal{H}_3 \ldots\ldots \rightarrow \mathcal{H}_\infty \tag{2.5}$$

An analysis of the parallelization efficiency of the successive approximations approach to the eigenvalue problem is presented in this paper. The refinement to an approximation may be applied independently on several processor nodes at the highest level of the algorithm [2], [3], [4], [5]. At the core of each successive approximations strand is a matrix diagonalization procedures has been well studied and displays high parallelization efficiencies [6], [7] (two-tiered approaches applied directly to the matrix diagonalization have been recently discussed [8]). In this paper, it is shown how both strategies can be simultaneously applied to successive approximation calculations. Within this approach, the two levels of parallelization are completely independent, allowing for analysis of a two-tiered parallelization extrapolated from data obtained from separate applications of the methods.

The literature on the title problem is vast. A series of publications (Lecissa. l978,1981. 1987) [9,10,11]listed hundreds of publications on the subject. Many of the previous studies concentrate on the theory of the subject. Obtaining natural frequencies only for those problems which permit exact solutions (Jones, 1973 ; Lin. 1974). Exact Navier-type solutions are possible for cross-ply plates having shear diaphragm boundaries (Jones, 1973) and antisymmetric angle-ply plates having a certain type of simple-support boundaries (S3). Exact Levy-type solutions are also possible for cross-ply laminates having two opposite shear diaphragm edges and for antisymmetric angle-ply laminate having two opposite S3 boundaries (Lin and King. 1979). Limited references are available on the study of the effects of many parameters like the material orthotropic characteristics, the number of layers, the lamination angle and boundary conditions on the natural frequencies of composite plates (Leissa and Naritn, 1989).

2.4 Fatigue Life Models

In these type of models, input information is taken from the *S-N* curves and a fatigue failure criterion is proposed. They do not take into account damage accumulation but predict the number of cycles at which fatigue failure occurs under fixed loading conditions.

One of the first fatigue failure criteria for composites was proposed by Hashin and Rotem[12,13]. They distinguished a fiber-failure and a matrix-failure mode the equations of which are as follows:

$$\sigma_L = X_f$$

$$\left(\frac{\sigma_T}{Y_f}\right)^2 + \left(\frac{\tau_{LT}}{S_f}\right)^2 = 1 \tag{2.6}$$

where σ_L and σ_T are the stresses along the fiber and transverse to the fiber, τ_{LT} is the shear stress and X_f, Y_f and S_f are the ultimate longitudinal tensile, transverse tensile and shear stress fatigue strength respectively. Since the ultimate strength is a function of fatigue stress level, stress ratio and number of cycles, the criterion is expressed in terms of three *S-N* curves which must be determined experimentally by testing on-axis unidirectional specimens under uni-axial load. This criterion is applicable for unidirectional ply and also if the failure modes are clearly distinguishable.

Whitworth[15] proposed a model to predict the fatigue life of composite specimens. He assumed that stress-strain response remains linear during fatigue cycle till fracture. The model is given as,

$$N_f = \exp\left[\frac{1}{h}\left\{\left(\frac{C_1 X_t}{\sigma_{\max}}\right)^{m/C_2} - 1\right\}\right] - 1 \tag{2.7}$$

where C_1, C_2 are constants to be determined experimentally and h and m are parameters that depend on applied stresses.

Ellyin and El-Kadi[16] demonstrated that the strain energy density can be used in a fatigue failure criterion for fiber-reinforced materials. The fatigue life N_f was related to the total energy input ΔW through a power law type relation of the form:

$$\Delta W = k N_f^{\alpha} \tag{2.8}$$

where k and α were shown to be functions of the fiber orientation angle. With the help of experimental data from tests on glass/epoxy specimens, an expression for α and k as a function of the fiber orientation angle was established. The strain energy density was calculated under an elastic plane stress hypothesis.

Philippidis and Vassilopoulos[17] proposed a multi-axial fatigue failure criterion, which is very similar to the well known Tsai-Wu quadratic failure criterion for static loading:

$$F_{ij}\sigma_i\sigma_j + F_i\sigma_i - 1 \le 0 \qquad \text{i, j = 1, 2, 6} \tag{2.9}$$

where F_{ij} and F_i are functions of the number of cycles N_f, the stress ratio R, and the frequency of loading v. The values of the static failure stresses X_t , X_c , Y_t , Y_c , and S for the calculation of the tensor components F_{ij} and F_i have been further replaced by the S-N curve values of the laminate along the same direction and under the same conditions. Although, in doing so, five S-N curves are required but the number was reduced to three due to assumption that $X_t = X_c$ and $Y_t = Y_c$. The authors used laminate properties to predict the laminate behavior, as they state that this enhances the applicability of the criterion to any stacking sequence of any type of composite (eg, unidirectional, woven, or stitched layers). This is because the *S-N* curves for the laminate account for the different damage types occurring in these various types of composite materials.

Fawaz and Ellyin[18] proposed a semi-log linear relationship between applied cyclic stress σ^c and the number of cycles to failure N_f:

$$\sigma^c = m\log(N_f) + b \tag{2.10}$$

The relation between the two sets of material parameters (m,b) and (m_r ,b_r) is specified by:

$$\begin{aligned} m &= f(a_1, a_2, \theta).g(R).m_r \\ b &= f(a_1, a_2, \theta).b_r \end{aligned} \tag{2.11}$$

Where a_1 is the first biaxial ratio $\left(a_1 = {}^{\sigma_y}\!/\!_{\sigma_x}\right)$, a_2 is the second biaxial ratio $\left(a_2 = {}^{\tau_{xy}}\!/\!_{\sigma_x}\right)$, R is the stress ratio and θ is the stacking angle. Their model could be generalized in the expression:

$$\sigma^c(a_1, a_2, \theta, R, N_f) = f(a_1, a_2, \theta).\left[g(R).m_r\log(N_f) + b_r\right] \tag{2.12}$$

The aim of the model was to predict the parameters *m* and *b* (related with m_r and b_r through the functions *f* and *g*) of a general *S*-log (N_f) line, for any *a*, θ, and *R*.

Philippidis and Vassilopoulos[17] compared their own results against the above-mentioned fatigue failure criterion proposed by Fawaz and Ellyin[18]. They concluded that the criterion by Fawaz and Ellyin was very sensitive to the choice of the reference *S-N* curve and that the predictions for tension-torsion fatigue of cylindrical specimens were not accurate. Under multi-axial loading the model by Philippidis and Vassilopoulos[17] can produce acceptable fatigue failure loci for all the data considered, but their choice of a multi-axial fatigue strength criterion based on the laminate properties implies that for each laminate stacking sequence a new series of experiments is required.

CHAPTER 3

DETERMINATION OF PROPERTIES OF LAMINA BY METHOD OF CELLS

3.0 Introduction

The method of cells is a micromechanical model which has been shown to accurately predict the overall behavior of various types of composites from the knowledge of the constituent properties. In particular, the method yields explicit effective constitutive equations for the inelastic behavior of metal matrix composites

The overall behavior of inelastic, multi-phase, unidirectional fibrous composites generated by the method of cells from the knowledge of the properties of the individual constituents is displayed in terms of:

- Effective elastic moduli
- Effective coefficients of thermal expansion
- Effective thermal conductivities
- Effective stress-strain response in the inelastic region

In the generalized formulation, the repeating unit cell is subdivided into an arbitrary number of subcells. This generalization extends the modelling capability of the method of cells to include the following:

- Thermo mechanical response of multi-phase, metal matrix composites
- Modeling of variable fiber shapes
- Analysis of different fiber arrays
- Modeling of porosities and damage
- Modeling of interfacial regions around inclusions, including interracial degradation

The continuum model for unidirectional fiber-reinforced materials is based on the assumption that the continuous square fibers extend in the x1 direction and are arranged in a doubly periodic array in the x2 and x3 directions, fig. 3.1. The cross section of the square fiber is h_1^2 and h2 represents its spacing in the matrix. As a result of this periodic arrangement, it is sufficient to analyze a representative cell as in fig. 3.2. the representative cell contains four subcells β,γ=1,2. Let four local coordinate systems ($\bar{x}_1, \bar{x}_2^{\beta}, \bar{x}_3^{\gamma}$) be introduced, all of which have origins that are located at the centre of each subcell.

Jacob Aboudi and Marek-Jerzy Pindera [19]

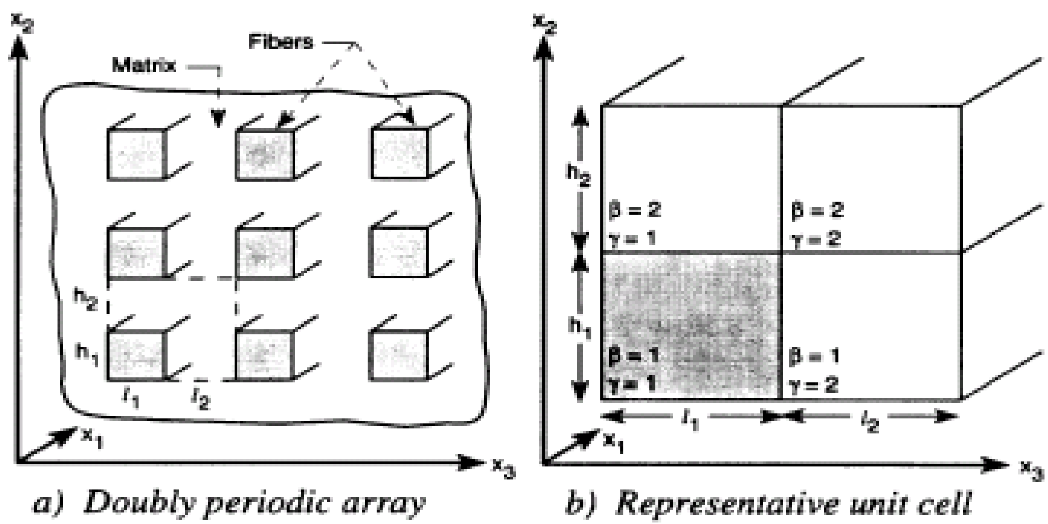


Fig. 3.1 Doubly-periodic Array And a Repeating Unit Cell For The Original Method Of Cells.

3.1 Assumptions

- Fiber arranged in periodic manner to form periodic array.
- Continuous square fiber as the area of circle and square is same.
- Fiber and matrix are perfectly elastic materials.
- Plane stress condition while calculating strength.
- Eliminating the micro-variables in average heat flux calculation.

3.2 Mathematical Model

Jacob Aboudi and Marek-Jerzy Pindera [19]

3.2.1 Calculation of Elastic Properties of Lamina

(E1, E2, G12, ν12, ν21)

A). The first order displacement expansion in each subcell is given as

$$u_i^{(\beta\gamma)} = w_i^{(\beta\gamma)} + \bar{x}_2^{(\beta)}\phi_i^{(\beta\gamma)} + \bar{x}_3^{(\gamma)}\varphi_i^{(\beta\gamma)} \quad i = 1,2,3. \tag{3.01}$$

Where $w_i^{(\beta\gamma)}$ -- Displacement component of the centre of the subcell

$\phi_i^{(\beta\gamma)}, \varphi_i^{(\beta\gamma)}$ – Characterize the linear dependence of the displacements in the local coordinates $\bar{x}_2^{(\beta)}, \bar{x}_3^{(\gamma)}$

The components of the small strain tensor are given as

$$\varepsilon_{ij}^{(\beta\gamma)} = \frac{1}{2}\left[\partial_i u_j^{(\beta\gamma)} + \partial_j u_i^{(\beta\gamma)}\right] \quad i, j = 1,2,3. \tag{3.02}$$

Where $\partial_1 = \partial/\partial x_1, \partial_2 = \partial/\partial \bar{x}_2^{(\beta\gamma)}$ and $\partial_3 = \partial/\partial \bar{x}_3^{(\beta\gamma)}$

The stress are related to strains in the form

$$\sigma^{(\beta\gamma)} = C^{(\beta\gamma)} \in^{(\beta\gamma)} \tag{3.03}$$

The average stresses $\bar{\sigma}_{ij}$ in the composite are determined from the average stresses in the subcells $\bar{\sigma}_{ij}^{(\beta\gamma)}$, in the form

$$\bar{\sigma}_{ij} = \frac{1}{(A')} \sum_{\beta,\gamma=1}^{2} (a')^{\beta\gamma} \bar{\sigma}_{ij}^{(\beta\gamma)} \tag{3.04}$$

Where $(a')^{\beta\gamma} = h_\beta h_\gamma$, $(A') = (h_1 + h_2)^2$ which is the area representative cell and

$$\bar{\sigma}_{ij}^{(\beta\gamma)} = \frac{1}{(a')^{(\beta\gamma)}} \int_{-h_\beta/2}^{h_\beta/2} \int_{-h_\gamma/2}^{h_\gamma/2} \sigma_{ij}^{(\beta\gamma)} d\bar{x}_2^{(\beta)} d\bar{x}_3^{(\gamma)}$$

The condition for the continuity of tractions, which are imposed along the interfaces of the subcells of the representative cell in the average sense, lead to

$$\bar{\sigma}_{2i}^{(1\gamma)} = \bar{\sigma}_{2i}^{(2\gamma)}.$$
$$\bar{\sigma}_{3i}^{(\beta 1)} = \bar{\sigma}_{3i}^{(\beta 2)}. \tag{3.05}$$

B). Displacement interfacial conditions

At any instant, the normal and tangential displacements should be continuous at the interfaces of the subcells of the representative cell of fig. 3.2. It follows that

$$u_i^{(1\gamma)}\Big|_{\bar{x}_2^{(1)}=\mp h_1/2} = u_i^{(2\gamma)}\Big|_{\bar{x}_2^{(2)}=\pm h_2/2} \tag{3.06}$$

$$u_i^{(\beta 1)}\Big|_{\bar{x}_3^{(1)}=\pm h_1/2} = u_i^{(\beta 2)}\Big|_{\bar{x}_3^{(2)}=\mp h_2/2} \tag{3.07}$$

The $\pm$ signs denote the two different relations obtained, depending on whether the interface follows the subcell (1γ) or (2γ) and same other equation.

Imposing the continuity conditions at the interfaces between the subcells. The average strains in the subcells can be obtained as

$$\bar{\epsilon}_{11}^{(\beta\gamma)} = \frac{\partial}{\partial x_1} w_1,$$
$$\bar{\epsilon}_{22}^{(\beta\gamma)} = \phi_2^{(\beta\gamma)},$$
$$\bar{\epsilon}_{33}^{(\beta\gamma)} = \varphi_3^{(\beta\gamma)},$$
$$2\,\bar{\epsilon}_{12}^{(\beta\gamma)} = \phi_1^{(\beta\gamma)} + \frac{\partial}{\partial x_1} w_2,$$
$$2\,\bar{\epsilon}_{13}^{(\beta\gamma)} = \varphi_1^{(\beta\gamma)} + \frac{\partial}{\partial x_1} w_3,$$
$$2\,\bar{\epsilon}_{23}^{(\beta\gamma)} = \phi_3^{(\beta\gamma)} + \varphi_2^{(\beta\gamma)}. \tag{3.08}$$

The average strains in the composite are given as

$$\bar{\in}_{ij} = \frac{1}{(\mathrm{A}')} \sum_{\beta,\gamma=1}^{2} (\mathrm{a}')^{(\beta\gamma)} \, \bar{\in}_{ij}^{(\beta\gamma)} \,. \tag{3.09}$$

And we get

$$\bar{\in}_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right). \tag{3.10}$$

And we get the relationships between the average stresses and micro-variables, in the subcells of the representative cell,

$$\bar{\sigma}_{11}^{(\beta\gamma)} = C_{11}^{(\beta\gamma)} \,\bar{\in}_{11} + C_{12}^{(\beta\gamma)}\left(\phi_2^{(\beta\gamma)} + \varphi_3^{(\beta\gamma)}\right)$$

$$\bar{\sigma}_{22}^{(\beta\gamma)} = C_{12}^{(\beta\gamma)} \,\bar{\in}_{11} + C_{22}^{(\beta\gamma)}\phi_2^{(\beta\gamma)} + C_{23}^{(\beta\gamma)}\varphi_3^{(\beta\gamma)}$$

$$\bar{\sigma}_{33}^{(\beta\gamma)} = C_{12}^{(\beta\gamma)} \,\bar{\in}_{11} + C_{23}^{(\beta\gamma)}\phi_2^{(\beta\gamma)} + C_{22}^{(\beta\gamma)}\varphi_3^{(\beta\gamma)}$$

$$\bar{\sigma}_{12}^{(\beta\gamma)} = C_{44}^{(\beta\gamma)}\left(\frac{\partial w_2}{\partial x_1} + \phi_1^{(\beta\gamma)}\right)$$

$$\bar{\sigma}_{13}^{(\beta\gamma)} = C_{44}^{(\beta\gamma)}\left(\frac{\partial w_3}{\partial x_1} + \varphi_1^{(\beta\gamma)}\right) \tag{3.11}$$

$$\bar{\sigma}_{23}^{(\beta\gamma)} = C_{66}^{(\beta\gamma)}\left(\phi_3^{(\beta\gamma)} + \varphi_2^{(\beta\gamma)}\right).$$

C) Composite constitutive relations – square symmetry

In the following, the constitutive equations of the unidirectional composite are derived in explicit form. In these equations the average stresses and strains are related by closed form expressions.

a) Average normal stress-strain relations

For that at i=2 we have

$$\phi_2^{(12)} = \left(h\,\bar{\in}_{22} - h_2\phi_2^{(22)}\right)\Big/ h_1 \,,$$

$$\phi_2^{(21)} = \left(h\,\bar{\in}_{22} - h_1\phi_2^{(11)}\right)\Big/ h_2 \,,$$

Where $h = h_1 + h_2$. Similarly with i = 3 gives

$$\varphi_3^{(12)} = \left(h \in_{33} - h_2 \varphi_3^{(11)}\right) \Big/ h_2 ,$$

$$\varphi_3^{(21)} = \left(h \in_{33} - h_1 \varphi_3^{(22)}\right) \Big/ h_1 .$$

Then we solve equation for micro variables, the average normal stresses can be computed and obtain as

$$\bar{\sigma}_{11} = b_{11} \bar{\in}_{11} + b_{12} \bar{\in}_{22} + b_{13} \bar{\in}_{33}$$

$$\bar{\sigma}_{22} = b_{12} \bar{\in}_{11} + b_{22} \bar{\in}_{22} + b_{23} \bar{\in}_{33}$$

$$\bar{\sigma}_{33} = b_{13} \bar{\in}_{11} + b_{23} \bar{\in}_{22} + b_{33} \bar{\in}_{33} \quad (3.12)$$

These are the requested constitutive relations for the normal stresses and strain of the unidirectional composite. The coefficients b_{ij} are the effective elastic constants of the composite, and are given as

$$b_{11} = \left[(a')^{11} C_{11}^{f} + C_{11}^{m} \left((a')^{12} + (a')^{21} + (a')^{22} \right) + \left(C_{12}^{m} - C_{12}^{f} \right)(Q_2 + Q_3) \right] \Big/ \mathrm{A}' ,$$

$$b_{12} = \left[(h/h_1)\left(C_{12}^{m} (a')^{12} + Q_1 C_{22}^{m} + Q_3 C_{23}^{m} \right) + (h/h_2)\left(C_{12}^{m} (a')^{21} + Q_2 C_{22}^{m} + Q_4 C_{23}^{m} \right) \right] \Big/ \mathrm{A}' ,$$

$$b_{13} = \left[(h_4/h_1)\left(C_{12}^{m} (a')^{12} + Q_1'' C_{22}^{m} + Q_3'' C_{23}^{m} \right) + (h_4/h_3)\left(C_{12}^{m} (a')^{21} + Q_2'' C_{22}^{m} + Q_4'' C_{23}^{m} \right) \right] \Big/ \mathrm{A}' ,$$

$$b_{22} = \left[(h/h_1)\left[C_{22}^{m} \left((a')^{12} + Q_1' \right) + Q_3' C_{23}^{m} \right] + (h/h_2)\left[C_{22}^{m} \left((a')^{21} + Q_2' \right) + Q_4' C_{23}^{m} \right] \right] \Big/ \mathrm{A}' ,$$

$$b_{23} = \left[(h/h_1)\left[C_{23}^{m} \left((a')^{21} + Q_2' \right) + Q_4' C_{22}^{m} \right] + (h/h_2)\left[C_{23}^{m} \left((a')^{12} + Q_1' \right) + Q_3' C_{22}^{m} \right] \right] \Big/ \mathrm{A}' ,$$

$$b_{33} = \left[(h_4/h_1)\left[C_{22}^{m} \left((a')^{12} + Q_1''' \right) + Q_3''' C_{23}^{m} \right] + (h_4/h_3)\left[C_{22}^{m} \left((a')^{21} + Q_2''' \right) + Q_4''' C_{23}^{m} \right] \right] \Big/ \mathrm{A}'$$

(3.13 --- 3.18)

b) The average axial shear stress-strain relations

With i = 1 we have

$$\phi_1^{(21)} = \left(h \frac{\partial w_1}{\partial x_2} - h_1 \phi_1^{11} \right) \Big/ h_2 ,$$

$$\phi_1^{(12)} = \left(h \frac{\partial w_1}{\partial x_2} - h_2 \phi_1^{22} \right) \Big/ h_1 ,$$

$$\phi_1^{(11)} = \left(h \frac{\partial w_1}{\partial x_2} C_{44}^m - \frac{\partial w_2}{\partial x_1} h_2 \left(C_{44}^f - C_{44}^m \right) \right) \Big/ \Delta,$$

$$\phi_1^{(22)} = \frac{\partial w_1}{\partial x_2}.$$

$$\bar{\sigma}_{12} = 2b_{44} \in_{12},$$

$$\bar{\sigma}_{13} = 2b_{44} \in_{13},$$

$$\bar{\sigma}_{23} = 2b_{66} \in_{23}. \qquad (3.19)$$

The effective elastic axial shear modulus of the unidirectional composite:

$$b_{44} = C_{44}^m \left\{ C_{44}^f \left[h\left((a')^{11} + (a')^{21} \right) + h_2 \left((a')^{12} + (a')^{22} \right) \right] C_{44}^m \left((a')^{12} + (a')^{22} \right) h_1 \right\} \Big/ \left(\mathrm{A}'\Delta_1 \right)$$

$$b_{55} = C_{44}^m \left\{ C_{44}^f \left[h_4\left((a')^{11} + (a')^{21} \right) + h_3 \left((a')^{12} + (a')^{22} \right) \right] C_{44}^m \left((a')^{12} + (a')^{22} \right) h_1 \right\} \Big/ \left(\mathrm{A}'\Delta_2 \right)$$

$$b_{66} = C_{66}^f C_{66}^m h h_4 \Big/ \delta \qquad (3.20 \text{ --- } 3.23)$$

Where $\Delta_1 = h_1 C_{44}^m + h_2 C_{44}^f$ for $\Delta_2 \Rightarrow h_2 = h_3$

$$\delta = h_1^2 C_{66}^m + \left(h_1 h_2 + h_1 h_3 + h_2 h_3 \right) C_{66}^f$$

The constants used here given in appendix, so the elastic stiffness matrix B = [b_{ij}]:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{12} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{13} & b_{23} & b_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{66} \end{bmatrix} \qquad (3.24)$$

The composite constitutive relations are of the form

$$\bar{\sigma} = B \in \qquad (3.25)$$

This representation effectively provides an orthotropic material with a square symmetry.

D) The effective elastic constants of transverse isotropic material can be determined from

$$E = \frac{1}{\pi}\int_0^{\pi} B'(\xi)\, d\xi. \tag{3.26}$$

Where $B' = \left[b'_{ij}\right]$ obtain by transformation i.e. by rotating the x_1, x_2, x_3 coordinates around x_1-axis by an angle $'\xi'$.

Which provides the elastic stiffness matrix E = $[e_{ij}]$ in the form

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0 \\ e_{12} & e_{22} & e_{23} & 0 & 0 & 0 \\ e_{13} & e_{23} & e_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{66} \end{bmatrix} \tag{3.27}$$

$$e_{11} = b_{11} \tag{3.28}$$

$$e_{12} = e_{13} = b_{12} \tag{3.29}$$

$$e_{22} = e_{33} = (3/4)\,b_{22} + (1/4)\,b_{23} + (1/2)\,b_{66} \tag{3.30}$$

$$e_{23} = (1/4)\,b_{22} + (3/4)\,b_{23} - (1/2)\,b_{66} \tag{3.31}$$

$$e_{44} = e_{55} = b_{44}$$

(3.32)

$$e_{66} = (1/2)(e_{22} - e_{23}) \tag{3.33}$$

D.1) Compliance matrix calculation S_{ij} when n = = 1

$$S_{ij} = \left(e_{ij}\right)^{-1}$$

D.2) Compliance matrix calculation S*ij*

when n ! = 1 then no need to calculate e_{ij}

$$S_{ij} = \left(b_{ij}\right)^{-1}$$

E) Final elastic properties of lamina

$$E_1 = 1/S_{11} \qquad E_2 = 1/S_{22}$$
$$E_3 = 1/S_{33} \qquad v_{12} = -S_{12}E_1$$
$$v_{13} = -S_{13}E_1 \qquad v_{23} = -S_{23}E_2$$
$$G_{12} = 1/S_{44} \qquad G_{13} = 1/S_{55}$$
$$G_{23} = 1/S_{66} \qquad (3.34—3.42)$$

3.2.2 Calculation of Coefficient of Thermal Expansion (α_i)

The effective coefficient of thermal expansion of a unidirectional composite in the axial and transverse directions can be readily obtained given by Aboudi[20]

$$\left[\alpha_1, \alpha_2, \alpha_3, 0, 0, 0\right] = B^{-1}\Gamma \quad \text{where } \Gamma\left({}^{\text{o}}\,\text{kPa}\right) = \left(\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{12}, \Gamma_{13}, \Gamma_{23}\right). \qquad (3.43)$$

It should be noted that determination of coefficient of thermal expansions using above equation is not affected by transformation.

$$\Gamma_{11} = \left(\left(\Gamma_{22}^{m} - \Gamma_{22}^{f}\right)\left(Q_2 + Q_3\right) + \left(a'\right)^{11}\Gamma_{11}^{f} + \left(\left(a'\right)^{12} + \left(a'\right)^{21} + \left(a'\right)^{22}\right)\Gamma_{11}^{m}\right)\Big/ \text{A}'$$

$$\Gamma_{22} = \left(\left(\Gamma_{22}^{m} - \Gamma_{22}^{f}\right)\left(Q'_2 + Q'_3\right) + \left(a'\right)^{11}\Gamma_{22}^{f} + \left(\left(a'\right)^{12} + \left(a'\right)^{21} + \left(a'\right)^{22}\right)\Gamma_{22}^{m}\right)\Big/ \text{A}'$$

$$\Gamma_{33} = \left(\left(\Gamma_{22}^{m} - \Gamma_{22}^{f}\right)\left(Q_2''' + Q_3'''\right) + \left(a'\right)^{11}\Gamma_{22}^{f} + \left(\left(a'\right)^{12} + \left(a'\right)^{21} + \left(a'\right)^{22}\right)\Gamma_{22}^{m}\right)\Big/ \text{A}'$$

$$\Gamma_{12} = 0, \Gamma_{13} = 0, \Gamma_{23} = 0$$

(3.43 --- 3.45)

And
$$\Gamma^n = \begin{pmatrix} C_{11}^n \alpha_l^n + 2C_{12}^n \alpha_t^n \\ C_{12}^n \alpha_l^n + \left(C_{22}^n + C_{23}^n\right)\alpha_t^n \\ C_{12}^n \alpha_l^n + \left(C_{22}^n + C_{23}^n\right)\alpha_t^n \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.46}$$

$$where \quad ij = 11, 22, 33, 12, 13, 23 = n = f, m$$
$$f = fiber, and \quad m = matrix$$

3.2.3 Calculation of Thermal Conductivity of Lamina

a) Fourier law for anisotropic material

$$q_i = -k_{ij}\left(\partial t \big/ \partial x_j\right) \tag{3.47}$$

Where $\boldsymbol{k_{ij}}$ Thermal conductivity tensor

b) Free expansion due to temperature difference is

$$\Delta\theta^{(\beta\gamma)} = \Delta T + \overline{x}_2^{(\beta)}\xi_2^{(\beta\gamma)} + \overline{x}_3^{(\gamma)}\xi_3^{(\beta\gamma)} \tag{3.48}$$

By applying continuity condition of temperature at the interfaces the average heat flux in the subcell is given as

$$\overline{q}_i^{(\beta\gamma)} = -k_l^{(\beta\gamma)}\,\partial T \big/ \partial x_j \quad i, j = 1, 2, 3 \tag{3.49}$$

c) The average heat flux in the composite is given as

$$\overline{q}_i = \frac{1}{A'}\sum_{i=1}^{n} a'_{\beta\gamma}\,\overline{q}_i^{(\beta\gamma)}$$

(3.50)

The continuity condition of the heat flux at the interface is given as

$$\overline{q}_i^{(1\gamma)} = \overline{q}_i^{(2\gamma)} \tag{3.51}$$

Where $\boldsymbol{i = 2, 3}$ and by eliminating the micro-variables

$$\overline{q}_j^{(\beta\gamma)} = -k_i^{(\beta\gamma)} \partial T \Big/ \partial x_j \quad i = l, t \ \& \ j = 1, 2, 3 \tag{3.52}$$

d) The effective (apparent) axial & transverse conductivity is

$$k_1 = \left((a')^{11} k_l^f + \left((a')^{12} + (a')^{21} + (a')^{22} k_l^m \right) \right) \Big/ h^2$$

$$k_2 = k_t^m \left\{ \begin{array}{l} k_t^f \left[h\left((a')^{11} + (a')^{21} \right) + h_2 \left((a')^{12} + (a')^{22} \right) \right] \\ + k_t^m \left((a')^{12} + (a')^{22} \right) h_1 \end{array} \right\} \Bigg/ \left[h^2 \left(k_t^f h_2 + k_t^m h_1 \right) \right]$$

$$k_3 = k_t^m \left\{ \begin{array}{l} k_t^f \left[h_4 \left((a')^{11} + (a')^{21} \right) + h_3 \left((a')^{12} + (a')^{22} \right) \right] \\ + k_t^m \left((a')^{12} + (a')^{22} \right) h_1 \end{array} \right\} \Bigg/ \left[h h_4 \left(k_t^f h_3 + k_t^m h_1 \right) \right]$$

(3.53—3.55)

3.2.4 Calculation of Strengths of Lamina

a. Calculating the stiffness matrices C_{ij}^n of lamina by using equations given appendix 'A'

b. Calculation of compliance matrices as

A. For fiber

$$S_{ij}^f = \left(C_{ij}^f \right)^{-1} \tag{3.56}$$

B. For matrix

$$S_{ij}^m = \left(C_{ij}^m \right)^{-1} \tag{3.57}$$

C. For lamina

$$S_{ij} = \left(C_{ij}\right)^{-1} \tag{3.58}$$

c. Calculation of stress concentration matrix

The stress concentration factors are taken into consideration to relate the overall composite stress to the average stress in the matrix phase as

$$\bar{\sigma}^{(ij)} = B^{(ij)}\bar{\sigma}. \tag{3.59}$$

for transversely isotropic composite the stress concentration matrix as per Aboudi[3] is

$$B = \begin{bmatrix} Bs_{11}^{ij} & Bs_{12}^{ij} & Bs_{12}^{ij} & 0 & 0 & 0 \\ Bs_{12}^{ij} & Bs_{22}^{ij} & Bs_{23}^{ij} & 0 & 0 & 0 \\ Bs_{12}^{ij} & Bs_{23}^{ij} & Bs_{22}^{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & Bs_{44}^{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & Bs_{44}^{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 & Bs_{66}^{ij} \end{bmatrix} \tag{3.60}$$

The elements of stress concentration matrix can be expressed explicitly in terms of the compliance elements of the composite, fiber and matrix phases and the volume fraction using Hill[5] relations

A. For fiber

$$Bs^{f} = \left(1/\mathrm{V}_{f}\right)\left(S_{ij}^{f} - S_{ij}^{m}\right)^{-1}\left(S_{ij} - S_{ij}^{m}\right) \tag{3.61}$$

B. For matrix

$$Bs^{ij} = \left(1/\mathrm{V}^{ij}\right)\left(S_{ij}^{m} - S_{ij}^{f}\right)^{-1}\left(S_{ij} - S_{ij}^{f}\right) \tag{3.62}$$

Now we have two 6x6 matrices Bs^{f} and Bs^{ij}

d. Calculation of strength of lamina

A. Ultimate Longitudinal stress of lamina

Consider a lamina subjected to unidirectional loading along fiber direction i.e. in direction 1. The lamina fails when fiber will fail, as stress concentration is more in fiber when subjected to loading in direction 1, so first element of stress concentration matrix of fiber is key element.

$$X_t = X_t^f \Big/ Bs_{11}^{11} \qquad X_c = X_c^f \Big/ Bs_{11}^{11} \tag{3.63}$$

B. Ultimate Transverse stress of lamina

In transverse direction matrix is weaker so stress concentration factor out of matrix subcells is maximum is the cause of failure of lamina. When that subcell fails, lamina will fail. There is transverse direction loading is there so second element of stress concentration matrix of matrix subcells is key factor.

$$Y_t = X_t^m \Big/ \left(\max_{(\beta\gamma)=(12),(21),(22)} \left(Bs_{22}^{(\beta\gamma)} \right) \right) \qquad Y_c = X_c^m \Big/ \left(\max_{(\beta\gamma)=(12),(21),(22)} \left(Bs_{22}^{(\beta\gamma)} \right) \right) \tag{3.64}$$

C. Ultimate Shear stress of lamina

In shear, lamina fails when shear stress in lamina matches with ultimate strength of matrix, so the forth element of stress concentration matrix of matrix subcells is key factor.

a) Calculation of shear strength S_{12} and S_{13}

$$S_{12} = S^m \Big/ \left(\max_{(\beta\gamma)=(12),(21),(22)} Bs_{44}^{(\beta\gamma)} \right)$$

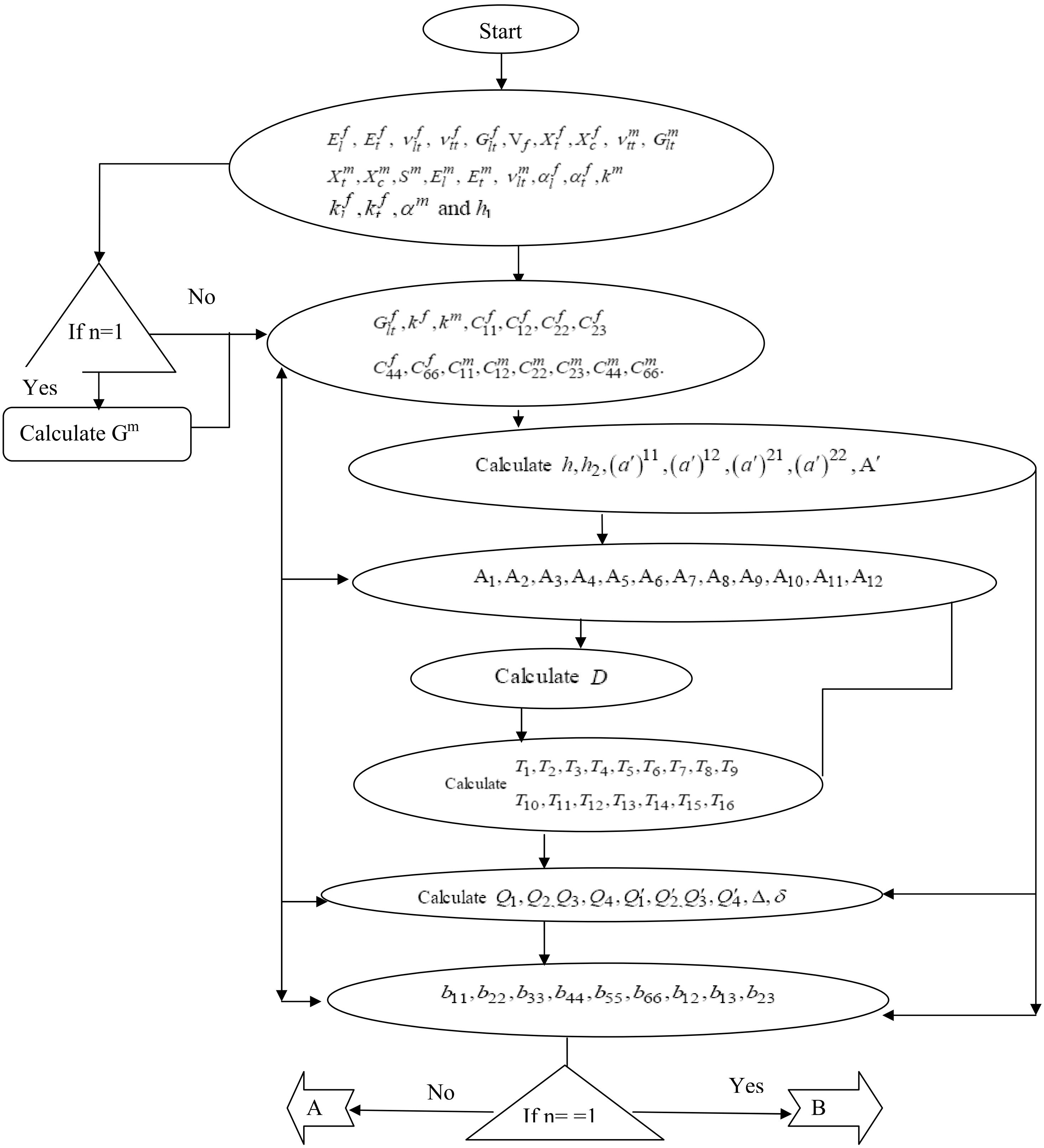
$$S_{13} = S^m \Big/ \left(\max_{(\beta\gamma)=(12),(21),(22)} Bs_{55}^{(\beta\gamma)} \right)$$

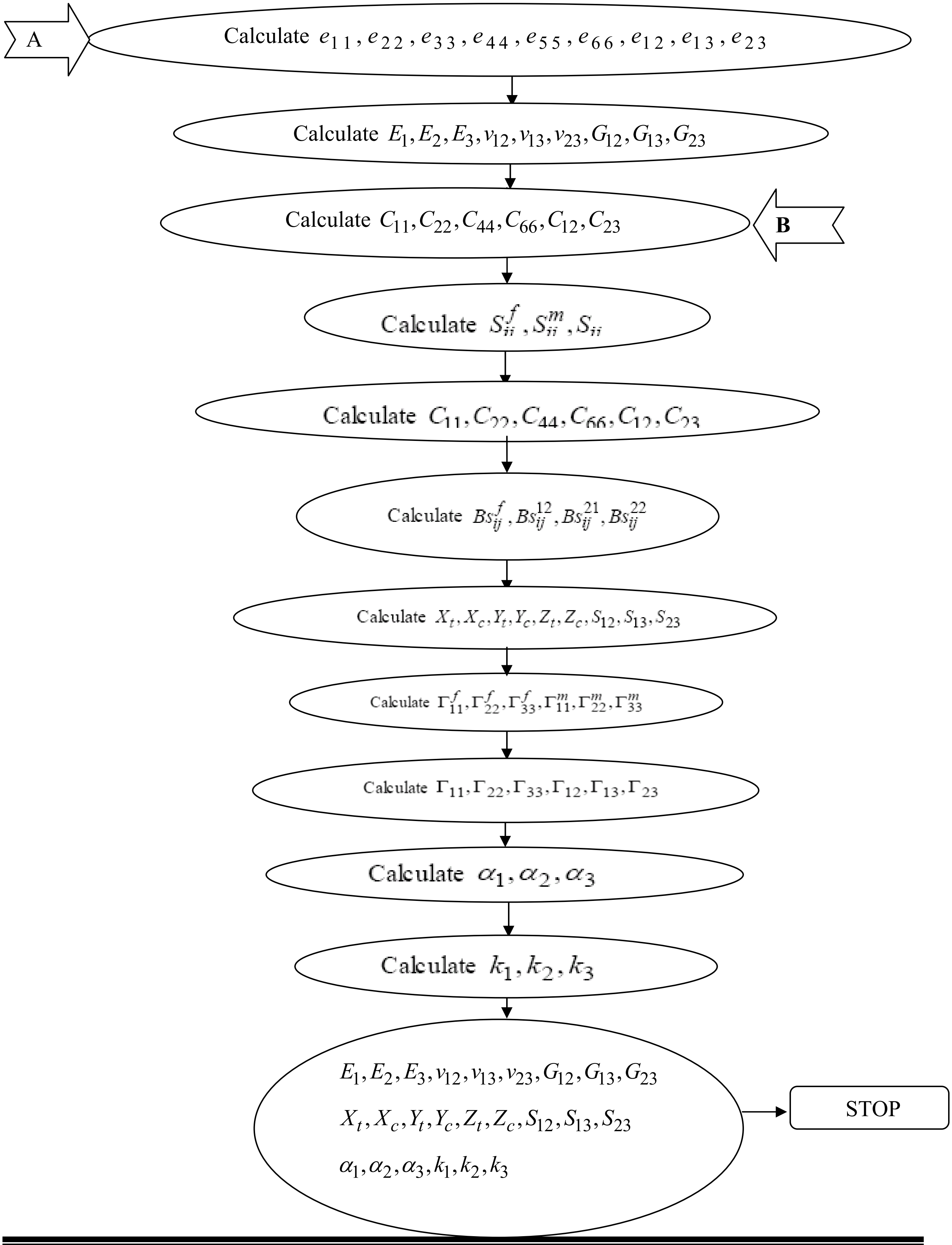
b) Calculation of shear strength S_{23}

$$Bs_{66}^{(\beta\gamma)} = \left(2C_{66}^{m} h h_4 C_{66}^{f} \left(1/G_{23} \right) \right) \Big/ \delta$$

$$S_{23} = S^m \Big/ \left(\max_{(\beta\gamma)=(12),(21),(22)} Bs_{66}^{(\beta\gamma)} \right)$$

3.3 Flow Chart For Calculating Strengths of Lamina




A
Calculate $e_{11}, e_{22}, e_{33}, e_{44}, e_{55}, e_{66}, e_{12}, e_{13}, e_{23}$
Calculate $E_1, E_2, E_3, v_{12}, v_{13}, v_{23}, G_{12}, G_{13}, G_{23}$
Calculate $C_{11}, C_{22}, C_{44}, C_{66}, C_{12}, C_{23}$
B
Calculate $S_{ii}^{f}, S_{ii}^{m}, S_{ii}$
Calculate $C_{11}, C_{22}, C_{44}, C_{66}, C_{12}, C_{23}$
Calculate $Bs_{ij}^{f}, Bs_{ij}^{12}, Bs_{ij}^{21}, Bs_{ij}^{22}$
Calculate $X_t, X_c, Y_t, Y_c, Z_t, Z_c, S_{12}, S_{13}, S_{23}$
Calculate $\Gamma_{11}^{f}, \Gamma_{22}^{f}, \Gamma_{33}^{f}, \Gamma_{11}^{m}, \Gamma_{22}^{m}, \Gamma_{33}^{m}$
Calculate $\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{12}, \Gamma_{13}, \Gamma_{23}$
Calculate $\alpha_1, \alpha_2, \alpha_3$
Calculate k_1, k_2, k_3
$E_1, E_2, E_3, v_{12}, v_{13}, v_{23}, G_{12}, G_{13}, G_{23}$
$X_t, X_c, Y_t, Y_c, Z_t, Z_c, S_{12}, S_{13}, S_{23}$
$\alpha_1, \alpha_2, \alpha_3, k_1, k_2, k_3$
STOP

3.4 Closure

Detailed discussion of mathematical formulation for all uni-axial lamina properties subjected to in-plane loading has been presented in this chapter. Required inputs for determination of the lamina properties by ―Method of Cells” are as follow

- Elastic properties of fiber and matrix.
- Ehermal properties of fiber and matrix.
- Strengths of fiber and matrix.

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CHAPTER 4

VIBRATION ANALYSIS OF COMPOSITES

4.0 Introduction

Laminated composite structures are becoming increasingly important in many engineering applications. The need for more information on the behavior of laminated structural components, like plates. is clear. Rectangular plates arc used in many engineering applications.

The use of vibration methods to obtain the elastic properties of materials appears to be getting more established. Apart from early work in this area , more recent contributions to theory and methodology continue to be made . The primary building block of laminated composite structures is the orthotropic lamina

Structural characteristics such as static deflections and bending stresses, buckling loads, and vibration frequencies are easily and exactly found for symmetrically laminated cross-ply plates when the fiber axes are parallel to the edges. However, for angle-ply plates the analysis is considerably more complicated and exact solutions are out of the question.

Currently, three separate, standard [21], static tests can yield the four independent elastic constants of the basic lamina (the Young's moduli in the longitudinal and transverse directions respectively, the inplane shear modulus, G_{12} , and the in-plane Poisson's ratio ν_{12}. However, all four may be obtained from a single experimental plate vibration test.

The three-dimensional stress-strain relationships for thick, orthotropic composite panels may be fully described by nine independent elastic constants. The appropriate engineering constants may be taken as E_1 , E_2 , and E_3 , the Young's moduli in the longitudinal transverse and thickness directions respectively, the shear moduli, G_{12} , G_{13} , and G_{23} , and the Poisson's ratios, ν_{12} , ν_{13} and ν_{23} across the three orthogonal planes. For transversely isotropic samples,(i.e., E2=E3 , G12=G13 and $\nu_{12}=\nu_{13}$), as is common with many types of composite materials, only five independent constants are needed.

The method that was developed by the first author and his colleagues [6-8] is applicable to the identification of any number of elastic constants of thin and thick plates, given enough vibration test data.

The Rayleigh-Ritz method with algebraic polynomial displacement functions is used to solve the vibration problem for laminated composite plates having different boundary conditions (The Rayleigh-Ritz method is used to yield a frequency equation, and the displacement function is expressed in algebraic polynomial).Natural frequencies for cantilever rectangular plates, plates having all its edges simply supported, plates having all edges free are presented .Convergence studies are made and reasonably accurate and comprehensive results were obtained.. The effect of various parameters (material, fiber orientation and boundary conditions) up on the natural frequencies is studies.

The literature on the title problem is vast. A series of publications (Lecissa. l978,1981. 1987) listed hundreds of publications on the subject. Many of the previous studies concentrate on the theory of the subject. Obtaining natural frequencies only for those problems which permit exact solutions (Jones, 1973 ; Lin. 1974). Exact Navier-type solutions are possible for cross-ply plates having shear diaphragm boundaries (Jones, 1973) and antisymmetric angle-ply plates having a certain type of simple-support boundaries (S3). Exact Levy-type solutions are also possible for cross-ply laminates having two opposite shear diaphragm edges and for antisymmetric angle-ply laminate having two opposite S3 boundaries (Lin and King. 1979). Limited references are available on the study of the effects of many parameters like the material orthotropic characteristics, the number of layers, the lamination angle and boundary conditions on the natural frequencies of composite plates (Leissa and Naritn, 1989).

It is shown that the energy functional derived are consistent with the equations of motion and boundary conditions and there fore can be used with energy approaches such as the Rayleigh-Ritz method. Thus equations were successfully-applied to obtain the natural frequencies of laminated composite plates. Rayleigh-Ritz with algebraic polynomial is used.

The primary purpose of the present work is to provide accurate and reasonably comprehensive results for the free-vibration frequencies of symmetrically laminated, simply supported plates, cantilever plates ,plates having all edges free, especially for lay-ups other than cross-ply (or specially orthotropic) for which no exact solutions are possible. For this purpose the Rayleigh-Ritz method is used with sufficient numbers of displacement function terms to obtain accurate results. Secondly, the effects of changing the numbers of layers, fiber orientation angles, material properties, and aspect ratios may be seen from the extensive results presented.

A complete and mathematically consistent set of equations, including equations of motion, boundary conditions and energy functionals is presented. It is shown that energy functionals derived there are consistent with the equations of motion and boundary conditions, and therefore can be used with energy approaches such as the Rayleigh-Ritz method.

In general situations, then, in seeking to determine the natural frequencies of laminated plates recourse must be made to approximate numerical methods. For single plates the traditional single field Rayleigh-Ritz method (RRM) can be employed if displacement fields can be proposed which are appropriate for the complete plate and which allow satisfaction of the relevant boundary conditions.

4.1 Analysis

The Rayleigh-Ritz method may be used to solve the free vibration problem. This utilizes the strain energy and kinetic energy functional for laminated plates. The strain energy stored in the plate during elastic deformation may be written in terms of the middle surface displacements u and v in directions tangential to the middle surface and parallel to the xz-and yz-planes, respectively, and the normal displacement w (see Fig. 4.2).

Arthur W. Leissa & Yoshihiro Narita[22]

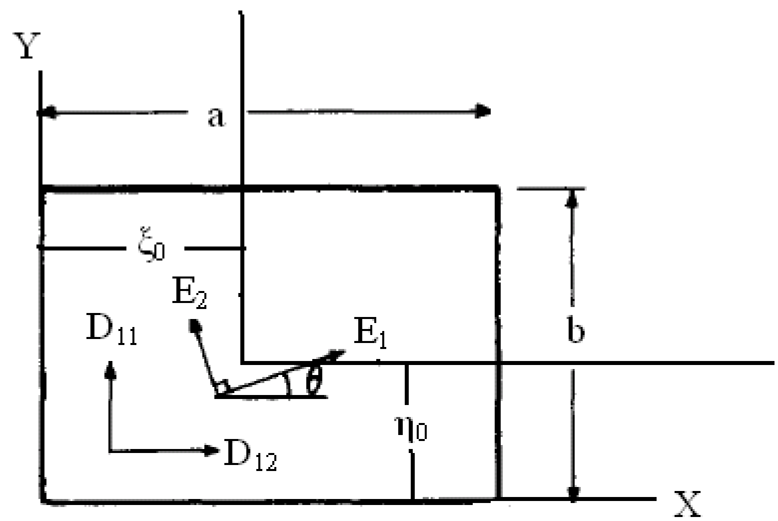

Fig. 4.2 Rectangular Plates With Coordinates

Consider a laminated composite rectangular plate of dimensions a x b as shown in Fig. 4.2. Each layer of the plate consists of parallel fibers bonded together by a matrix material. The fiber direction within a layer is indicated by the angle Θ in Fig. 4.2. The modulus of elasticity of the layer in the direction of the fibers is El, and the transverse modulus is E2.

The strain energy is (Qatu. 1989; Leissa and Qatu. 1991) [23,24,25]:

∂^2

(4.01)

Where the A_{ij} , B_{ij} and are the conventional laminated stiffness coefficients.

The above energy functional may be expressed in terms of middle surface strains and curvature changes which are related to the middle surface displacements by

$$\varepsilon_x = \frac{\partial u}{\partial x},$$

$$\varepsilon_y = \frac{\partial v}{\partial y},$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{4.02}$$

The total kinetic energy is

$$T = \frac{\rho}{2}\iint \{\dot{u}_0^2 + \dot{v}_0^2 + \dot{w}_0^2\}\, \partial x\, \partial y$$

(4.03)

Where ρ is the average mass density of the composite plate per unit volume.

4.1.1For Plates Having All Edges Free

Displacements are assumed as

$$\begin{aligned} u(x,y,t) &= U(x,y)\sin\omega t \\ v(x,y,t) &= V(x,y)\sin\omega t \\ w(x,y,t) &= W(x,y)\sin\omega t \end{aligned}$$

(4.04)

Algebraic trial functions will be used in the analysis because first. they do form a complete set of functions, which guarantees convergence to the exact solution as the number of terms taken increases and. second. one can straightforwardly solve for many boundary conditions using algebraic trial functions.

The displacement functions U, V and W can be written in terms of the non-dimensional coordinates ξ and η as :

$$U(\xi,\eta) = \sum_{i=i_0}^{I} \sum_{j=j_0}^{J} \alpha_{ij}\, \xi^i \eta^j$$

$$V(\xi,\eta) = \sum_{k=k_0}^{K} \sum_{l=l_0}^{L} \beta_{kl}\, \xi^k \eta^l$$

$$W(\xi,\eta) = \sum_{m=m_0}^{M} \sum_{n=n_0}^{N} \gamma_{mn}\, \xi^m \eta^n$$

(4.05)

where $\xi = \frac{x}{a} - \xi_0$, and $\eta = \frac{y}{b} - \eta_0$ and ξ_0 , η_0 are defined in Fig. 1.

Keeping in mind that the Rayleigh-Ritz method requires satisfaction of geometric (forced) boundary conditions only, with suitable selection of the values ξ_0 , η_0 , i_0 , j_0 , k_0 , l_0 , m_0 and n_0 one can solve for many boundary conditions with the same analytical procedure.

4.1.2 For Simply Supported Symmetrically Laminated Rectangular Plates

The transverse displacement w(x, y, t) of a plate vibrating freely may be written as

$$w(x, y, t) = W(x, y)\sin \omega t \tag{4.06}$$

The maximum strain energy (U) stored in the plate during bending in a vibratory cycle is given by Arthur W. Leissa & Yoshihiro Narita[22]

$$U = \frac{1}{2}\iint \left[D_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + D_{22}\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 4D_{16}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial x \partial y} + 4D_{66}\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right] \partial x \partial y \tag{4.07}$$

Where the D_{ij} are the well-known stiffness coefficients relating the moment resultants (M_x, M_y, M_{xy}) to the bending curvatures (K_x, K_y, K_{xy}) by the matrix relationship

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}\begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix} \tag{4.08}$$

The maximum kinetic energy during a vibratory cycle is

$$T = \frac{\rho\omega^2}{2}\iint W^2 \partial x \, \partial y \tag{4.09}$$

Transverse displacements are assumed in the form

$$W(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{4.10}$$

4.1.3 For Cantilever Laminated Plates

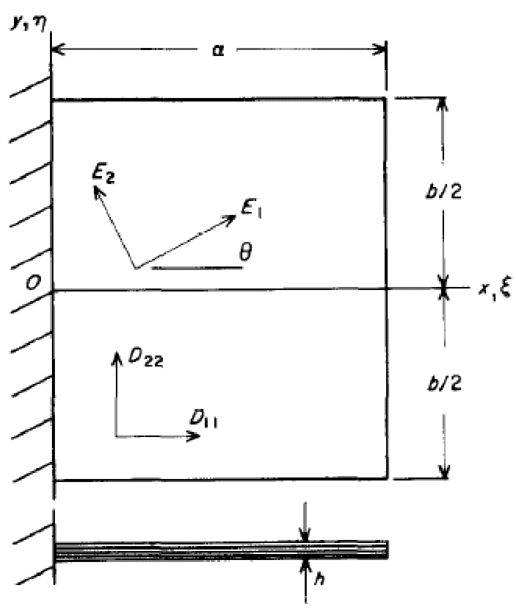

Fig 4.3 Laminated cantilever plate with co-ordinate conventions.

The strain energy stored in the shell during elastic deformation may be written in terms of the middle surface displacements u and v in directions tangent to the middle surface and parallel to the xz and yz planes, respectively, and the normal displacement w. It may be expressed as the sum of four parts

(Mohamad S. Qatu & Arthur W.)[26]

$$U = U_{or} + U_{es} + U_{bs} + U_{bt}$$

(4.11)

Where U_{or} includes terms due to orthotropic characteristics of the material (i.e. A_{11}, A_{12}, A_{22}, A_{66}, D_{11}, D_{12}, D_{22}, D_{66}).

$$U_{or} = 1/2 \iint \{A_{11} (\partial u/\partial x)^2 + A_{66} (\partial u/\partial y)^2 + A_{22} (\partial v/\partial y)^2 + A_{66} (\partial v/\partial x)^2 + [A_{11}/(R_x^2) + A_{2}$$

(4.12)

U_{es} includes terms of extension-shearing coupling (i.e. A 16, A26).

$$U_{es} = \frac{1}{2}\iint \Big\{2A_{16}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + 2A_{26}\left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) + 2A_{16}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right) + 2A_{26}\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + 2\left(\frac{A_{26}}{R_y} + \frac{A_{16}}{R_x}\right)w\frac{\partial v}{\partial x} + 2\Big(\frac{A_{26}}{R_y} + \frac{A_{16}}{R_x}$$

(4.13)

U_{bs} includes terms of bending-stretching coupling (i.e. B_{ij})

∂y

(4.14)

And U_{bt} includes terms of bending-twisting coupling (i.e. D16 , D26).

$$U_{bt} = \frac{1}{2}\iint \left\{4D_{16}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right) + 4D_{26}\left(\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)\right\} \partial x \partial y \quad (4.15)$$

Where the A_{ij}, B_{ij} and D_{ij} are the conventional laminate stiffness coefficients

Whitney and Vinson and Sierakowski [27,28]

The above energy functional may be expressed in terms of middle surface strains and curvature changes which are related to the middle surface displacements by

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{w}{R_x}, \quad \epsilon_x = \frac{\partial v}{\partial y} + \frac{w}{R_y}, \quad \epsilon_x = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{2w}{R_{xy}}$$

$$k_x = -\frac{\partial^2 w}{\partial x^2}, \quad k_y = -\frac{\partial^2 w}{\partial y^2}, \quad \tau = -2\frac{\partial^2 w}{\partial x \partial y} \quad (4.16)$$

The total kinetic energy is

$$T = \frac{\rho}{2}\iint \{\dot{u}_0^2 + \dot{v}_0^2 + \dot{w}_0^2\}\, \partial x\, \partial y \quad (4.17)$$

where ρ is the average mass density of the composite shell per unit volume.

For free vibrations of a shallow shell having the rectangular planform
shown in Fig. 4.2, displacements are assumed as

$$u(x, y, t) = U(x, y)\sin \omega t$$
$$v(x, y, t) = V(x, y)\sin \omega t$$
$$w(x, y, t) = W(x, y)\sin \omega t$$

(4.18)

Algebraic polynomial trial functions will be used in the analysis principally for two reasons. The first is that they form a mathematically complete set of functions, which guarantees convergence to the exact solution as the number of terms taken increases. 14 The second reason is that they are relatively simple to use in the algebraic manipulation and computer programming subsequently required, and can be differentiated and integrated exactly in the energy functional needed. Thus the displacement functions U, V and W are written in terms of the non -dimensional coordinates ξ and η as

$$U(\xi, \eta) = \sum_{i=1}^{I} \sum_{j=0}^{J} \alpha_{ij} \xi^{i} \eta^{j}$$

$$V(\xi, \eta) = \sum_{k=1}^{K} \sum_{l=0}^{L} \beta_{kl} \xi^{k} \eta^{l}$$

$$W(\xi, \eta) = \sum_{m=2}^{M} \sum_{n=0}^{N} \gamma_{mn} \xi^{m} \eta^{n}$$

(4.19)

where ξ =2x/a,η =2y/b, a and b are the platform dimensions (Fig. 4.2) and α_{ij} , β_{kl} and γ_{mn}, are arbitrary coefficients to be determined subsequently. It should be stated here that the Ritz method requires satisfaction of geometric (forced) boundary conditions only, and thus the indices in eqns

(4.22) begin with i = 1, k = 1 and m = 2 to guarantee satisfaction of the clamped boundary conditions at ξ =0 for all terms of the polynomials.

For solving the free vibration problem, eqn (4.04,4.21) and (4.05,4.22) are substituted into eqn (4.01,4.11) in order to get an expression for the maximum strain energy (U_{max}) and into eqn (4.03,4.20) in order to get an expression for the maximum kinetic energy (T_{max}). The Ritz method requires minimization of the functional (T_{max}- U_{max}) which can be accomplished by taking the derivatives:

$$\frac{\partial(T_{max}-U_{max})}{\partial\alpha_{ij}}=0 \qquad i=1,2,\ldots\ldots I \quad ; \quad j = 0,1\ldots\ldots J$$

$$\frac{\partial(T_{max}-U_{max})}{\partial\beta_{kl}}=0 \qquad k=1,2,\ldots\ldots K \quad ; \quad l = 0,1\ldots\ldots L$$

$$\frac{\partial(T_{max}-U_{max})}{\partial\gamma_{mn}}=0 \qquad m=2,3,\ldots\ldots M \quad ; \quad n = 0,1\ldots\ldots N \tag{4.22}$$

This yields a total of I x (J+I) + K x (L+I)+ (M-1) x (N+I) simultaneous, linear, homogeneous equations in an equal number of unknowns α_{ij} , β_{kl} and γ_{mn} .Those equations can be described

$$(K-\lambda^2 M)\alpha=0 \tag{4.23}$$

4.2 Flow Chart for Typical Successive Approximations Approach to The Eigen value Problem

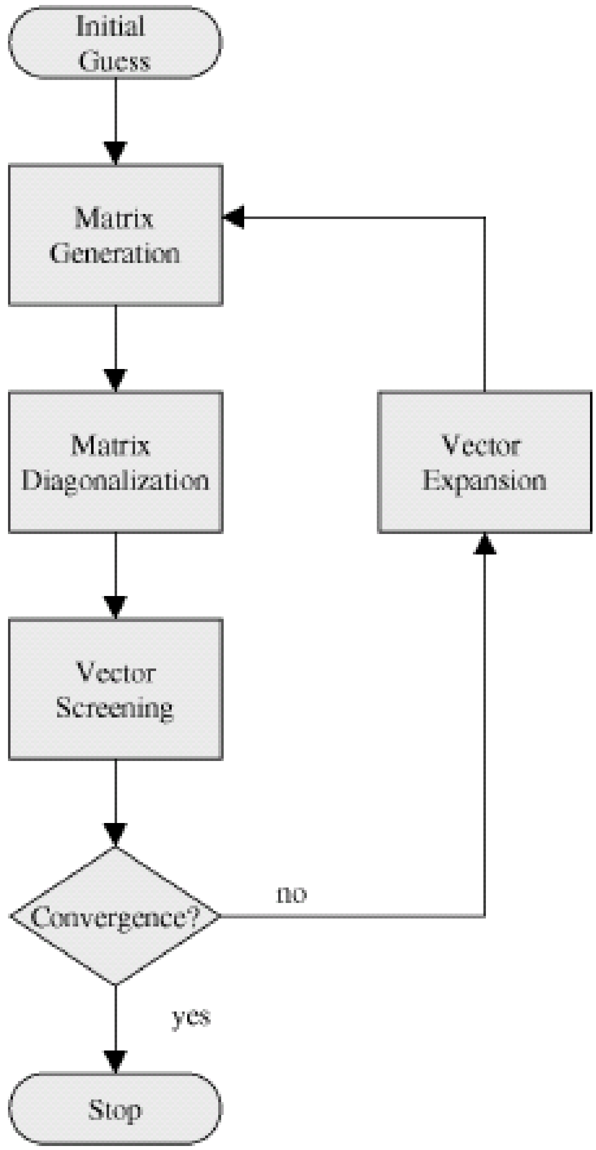

Fig 4.4 Typical Program Flow For a Successive Approximations Approach.

4.3 Closure

Comparisons among results from the present method and analytical and experimental data obtained for laminated cantilever plates may be found in the dissertation by Qatu (1989) and the work accomplished by Qatu and Leissa (199la) [23,24,25]. There, the natural frequencies obtained experimentally, and those obtained by using the finite element method are compared with those obtained by the present method. It was concluded that the present method yields a closer upper bound to the exact solution than the finite element results and is reasonably close to the frequencies obtained experimentally.

CHAPTER 5
FATIGUE ANALYSIS OF COMPOSITE MATERIALS

5.0 Introduction

The development of lightweight structures requires, among others, reliable design methods against cyclic fatigue. One approach is to design, build and test prototypes which certainly results in the highest possible reliability, but consumes considerable time and development cost. Therefore, fatigue life prediction is highly desirable as it allows the assessment of design alternatives in early stages of the development process and the integration into a CAE design environment, but most of all it does not necessarily require early design stage prototyping.

The fatigue design methodology of polymer matrix composites (PMC) compared to metallic structural materials is still lacking with regard to the availability of methods as well as the insufficiency of existing models to reliably assess the fatigue behaviour. The reason for this situation is the inherent inhomogeneity of PMC and the various distinguished damage mechanisms such as matrix cracking, fiber/matrix debonding, fiber fracture, interlaminar delamination, as well as their pronounced interactions which have to be taken into consideration.

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Reifsnider KL and Stinchcomb WW [29] represents a non-linear fatigue life prediction methodology for layered composites which accounts for fatigue damage initiation and growth as well as final failure. This method subdivides relevant parts of a fatigue loaded structure into sub-critical and critical elements where sub-critical elements encompass in particular matrix cracking causing stiffness reduction and stress redistribution within the laminate without directly leading to the total failure of the laminate Stress redistribution is modeled through appropriate stiffness degradation relations applying the classical laminated plate theory. Critical elements are increasingly stressed, suffer from a decrease of their residual strength with increasing cyclic loading and finally cause the total failure of the structure when the residual strength of the critical element is reduced to the equivalent stress correspondent to the instantaneous external load. This life prediction methodology may be regarded as the most complete for polymeric composites available to date.

5.1 Fatigue Failure Criterion for Lamina (off axis loading)

5.1.1 E-glass/epoxy

Hashin and Rotem[30] conducted experiments (for stress ratio R=0.1 and cyclic frequency 19 Hz) on E-glass/epoxy lamina under cyclic loading. Equation for S-N curve along the fiber direction based on their experimental data (equation derived by curve fitting) is given as,

$$X_f = X_t \; f_L(N_f) \tag{5.1}$$

$$f_L(N_f) = a - b \; log_{10} N_f$$

where X_f is fatigue strength and $f_L(N_f)$ is fatigue function along fiber direction, a and b are constant parameters and their values are given by Hashin and Rotem[24] as 1.123 and 0.11 respectively. Material properties of UD lamina (E-Glass/Epoxy) along fiber direction are as:

E_L = 55.8 GPa, X_t = 1260 MPa Poisson's ratio $\mu_{LT} = 0.285$

The S-N diagram (curve fit of experimental data) along fiber direction
under constant stress ratio of 0.1 and frequency 19 Hz is shown below in Fig. 5.1

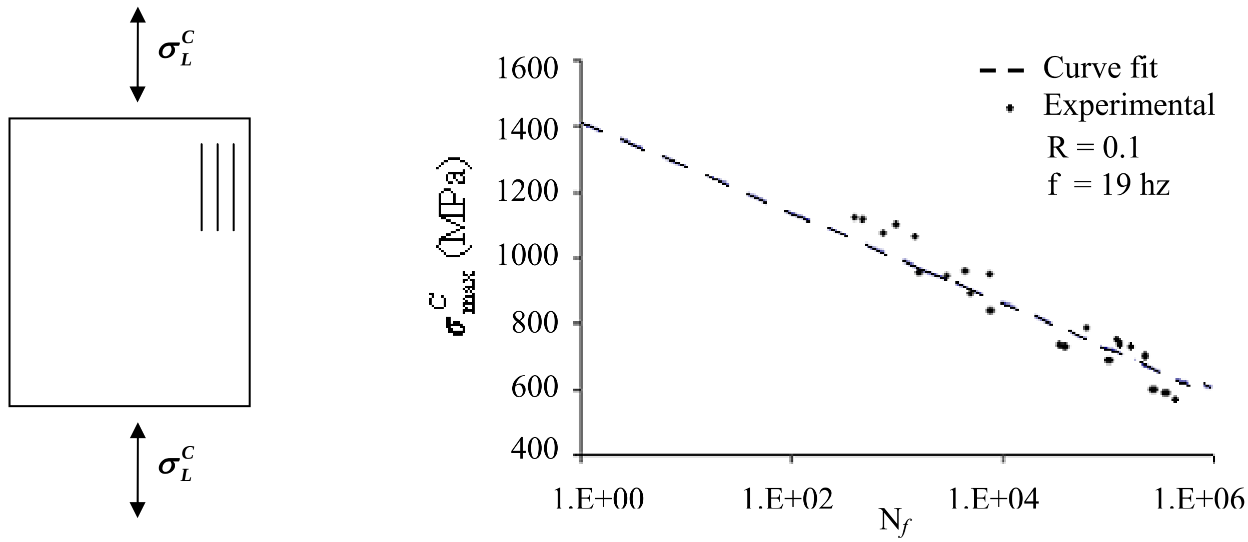


Fig. 5.1. Lamina Subjected To Fatigue Load Along Fiber Direction And Corresponding S-N Curve.

Equation for S-N curve along transverse to fiber direction based on experimental data (derived by curve fitting) is given by Hashin and Rotem[24] as,

$$Y_f = Y_t \ f_T(N_f) \tag{5.2}$$

$$f_T(N_f) = c - d \ log_{10} N_f$$

where Y_f is fatigue strength and $f_T(N_f)$ is fatigue function along transverse to fiber direction, c and d are constant parameters and their values are 0.956 and 0.0541 respectively, these parameters are developed by curve fitting[19]. The material properties for UD lamina E-glass/epoxy along transverse to fiber direction are as follows

E_T = 18.1 GPa $\qquad\qquad$ Y_t = 42 MPa.

S-N diagram (curve fit of experimental data) transverse to fiber direction under constant stress ratio of 0.1 and frequency 19 Hz is shown below in Fig. 5.2.

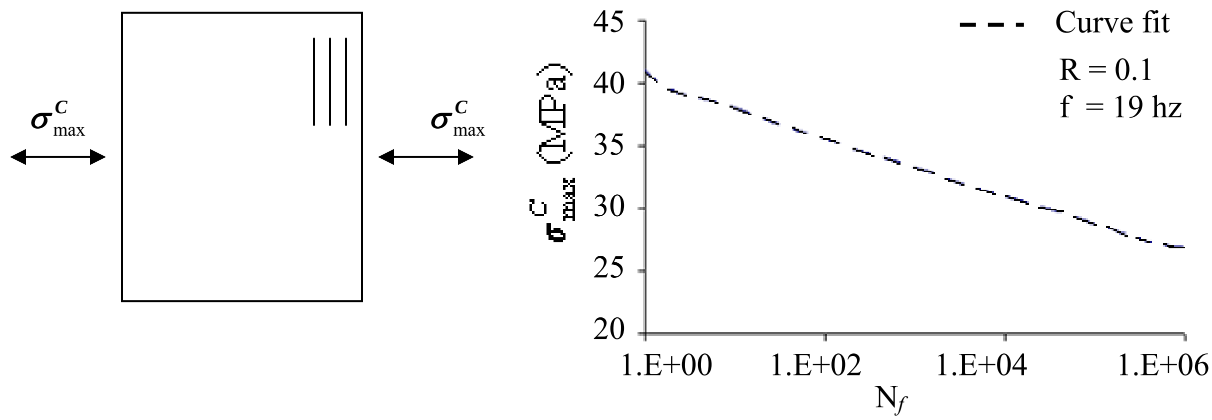


Fig. 5.2 Lamina Under Fatigue Load Along Transverse To Fiber Direction And Corresponding S-N Curve.

Equation for S-N curve in shear mode based on experimental data (derived by curve fitting) is given by Hashin and Rotem[25] as,

$$S_f = S\, f_\tau(N_f) \tag{5.3}$$

$$f_\tau(N_f) = g - h\ log_{10} N_f$$

where S_f is shear fatigue strength and $f_\tau(N_f)$ is fatigue function, g and h are constant parameters and their values are 0.917 and 0.0852 respectively. These parameters are found by curve fitting[24]. Shear modulus (G_{LT}) of UD lamina (E-glass/epoxy) is G_{LT} = 6.13 GPa and shear strength (S) is 84 MPa. S-N diagram (curve fit of experimental data) for shear mode under constant stress ratio 0.1 and frequency 19 Hz is shown below in Fig.5.3.

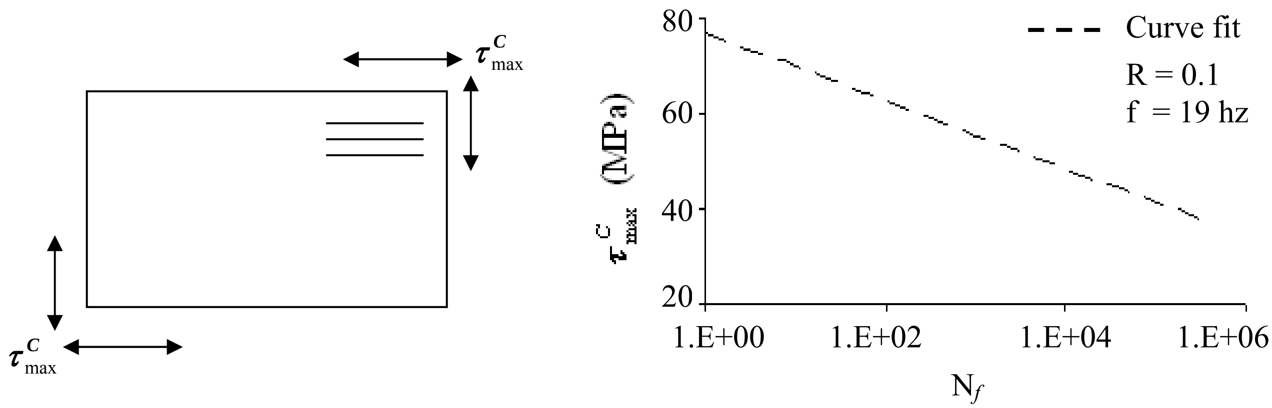


Fig. 5.3. Lamina Under Fatigue Load In Shear Mode And Corresponding S-N Curve.

5.1.2 Theory

Fatigue properties of a lamina, derived by experiments[24] are given in section 5.1.1 and 5.1.2. Our objective is to predict fatigue life of UD lamina subjected to off-axis loading as shown below in Fig. 5.4.

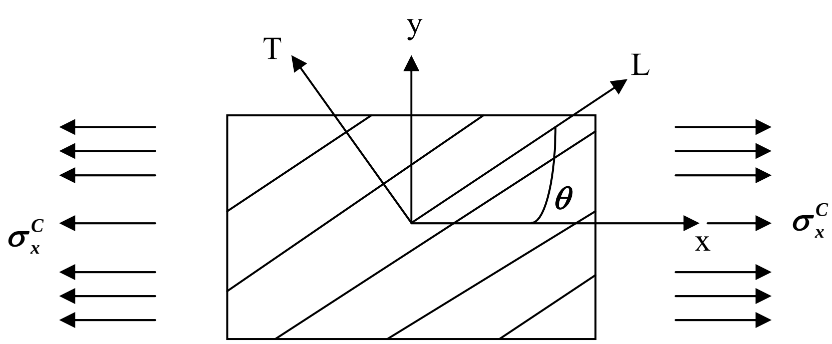


Fig. 5.4. Lamina Subjected To Off-axis Uniform Cyclic Loading At Angle.

Above figure shows uniform uniaxial cyclic stress applied on lamina along x-direction. By using transformation matrix the cyclic stress components in L, T, and in LT plane can be calculated as;

$$\begin{bmatrix} \sigma_L^c \\ \sigma_L^c \\ \tau_{LT}^c \end{bmatrix} = [T] \begin{bmatrix} \sigma_x^c \\ 0 \\ 0 \end{bmatrix}$$

(5.4)

where transformation matrix $[T] = \begin{bmatrix} cos^2\theta & sin^2\theta & 2\,sin\,\theta\,cos\,\theta \\ sin^2\theta & cos^2\theta & -2\,sin\,\theta\,cos\,\theta \\ -sin\,\theta\,cos\,\theta & sin\,\theta\,cos\,\theta & cos^2\theta\text{-}\,sin^2\theta \end{bmatrix}$

From equation 5.4 induced stresses in principal coordinates of lamina are given as

$$\begin{aligned} \sigma_L^c &= \sigma_x^c \ \ cos^2\theta \\ \sigma_T^c &= \sigma_y^c \ \ sin^2\theta \\ \tau_{LT}^c &= -\tau_{xy}^c \ \ cos\,\theta\,sin\,\theta \end{aligned} \tag{5.5}$$

Now using equation (5.1), (5.2), (5.3) and (5.5) in Tsai-Hill quadratic fatigue criteria as shown in equation (5.4), number of cycles to failure of lamina under off-axis uniform cyclic loading can be calculated.

$$\left(\frac{\sigma_L^c}{X_f}\right)^2 - \left(\frac{\sigma_L^c \sigma_T^c}{X_f X_f}\right) + \left(\frac{\sigma_T^c}{Y_f}\right)^2 + \left(\frac{\tau_{LT}^c}{S_f}\right)^2 = 1 \tag{5.6}$$

5.2 Generated Model

5.2.1 Objective

- Prediction of Fatigue behavior of composites using *S-N* curve approach.
- Determination of residual strength at any no. of cycles (n) and any R and f.
- Determination of fatigue life at any R and f.

5.2.2Assumptions

- When residual strength decreases to the maximum applied stress, fatigue failure occurs.
- The shape parameter (α),scale parameter (β) and are γ the material properties ,i.e. they do not depend on R and f.
- When residual strength decreases to the maximum applied stress, fatigue failure occurs.

The shape parameter (α),scale parameter (β) and are γ the material properties ,i.e. they do not depend on R and f.

5.2.3 Input Properties

S-N diagram of the laminate at some known

R and f $[\ R = \sigma_{min} / \sigma_{max}\ ]$

5.2.3 (a) Fatigue properties

Input S-N curve (3points on input S-N curve i.e.

$(\sigma_{max1}, \sigma_{max2}, \sigma_{max3}, N_{f1}, N_{f2}, N_{f3})$ at some known R and f

or

Equation for input S-N curve $(\sigma_{max} Vs\ N_f)$

5.2.3 (b) Static properties

X_T or X_C depending upon loading condition

5.3 Mathematical Derivation

Determination of fatigue properties from input S-N curve equation. We start with following deterministic equation for the rate of strength degradation

Jayantha A. Epaarachchi, Philip D. Clausen.[31]

$$\frac{dX_t^r}{dN} = -C\,N^{-m}$$

This equation can be converted to time-dependent equation as follows

$$\frac{dX_t^r}{dt} = -C_1\,t^{-m_1} \quad \ldots\ldots C1 = A_1 f(\sigma_{max}, R, X_T)$$

A_1 and m_1 are material constants

Integration of above equation yields

$$X_T^r = -A_1 f(\sigma_{max}, R, X_T)\; t^{-m1+1} / (-m_1+1) + \text{constant}$$

Putting

$$\alpha = -A_1 / (-m_1+1)$$

And $\beta = -m_1+1$ in above eq.

$$X_t^r = -\alpha f(\sigma_{\max}, R, X_t)\, t^{\beta} + constant$$

For constant frequency, $t = n / f$

Applying the boundary conditions

$$X_T^r = X_T \quad at \quad n = 1$$

$$X_T^r = \sigma_{max} \quad at \quad n = N_f$$

We get

$$X_T - \sigma_{max} = \alpha\, f(\sigma_{max}, R, X_T)\,(N_f^{\beta} - 1) / f^{\beta}$$

$$f(\sigma_{max}, R, X_T) = X_T^{\,1-\gamma}\, \sigma_{max}^{\gamma} (1-R)^{\gamma}$$

Putting the value of function f in origional eq.

$$X_T - \sigma_{max} = \alpha\, X_T^{\,1-\gamma}\, \sigma_{max}^{\gamma} (1-R)^{\gamma} (N_f^{\beta} - 1) / f^{\beta} \qquad (5.7)$$

Putting the values of the 3 points on the input S - N curve in the above equation i.e. putting σmax1 , σmax2 , σmax3 , Nf1 , Nf2 , Nf3 we get following 3 equations -

$$X_T - \sigma_{max1} = \alpha\, X_T^{\,1-\gamma}\, \sigma_{max1}^{\gamma} (1-R)^{\gamma} (N_{f1}^{\beta} - 1) / f^{\beta} \qquad (5.8)$$

$$X_T - \sigma_{max2} = \alpha\, X_T^{\,1-\gamma}\, \sigma_{max2}^{\gamma} (1-R)^{\gamma} (N_{f2}^{\beta} - 1) / f^{\beta} \qquad (5.9)$$

$$X_T - \sigma_{max3} = \alpha\, X_T^{\,1-\gamma}\, \sigma_{max3}^{\gamma} (1-R)^{\gamma} (N_{f3}^{\beta} - 1) / f^{\beta} \qquad (5.10)$$

These equations can be solved by the following method

Dividing eq. (5.8) by eq. (5.9)

$$(X_T - \sigma_{max1})/(X_T - \sigma_{max2}) = \sigma_{max1}^{\gamma}\ (N_{f1}^{\beta} - 1) / \sigma_{max2}^{\gamma} (N_{f2}^{\beta} - 1) \qquad (5.11)$$

Dividing eq. (5.8) by eq. (5.10)

$$(X_T - \sigma_{max1}\ /(X_T - \sigma_{max3}) = \sigma_{max1}^{\gamma}(N_{f1}^{\beta} - 1)/\sigma_{max3}^{\gamma} (N_{f3}^{\beta} - 1) \qquad (5.12)$$

Taking logarithm of both eq. (5) & eq. (6)

$$\log (X_T - \sigma_{max1}) - \log (X_T - \sigma_{max2}) = \gamma(\log\sigma_{max1} - \log\sigma_{max2}) + \log(N_{f1}^{\beta}-1) - \log(N_{f2}^{\beta}-1) \quad (5.13)$$

and

$$\log (X_T - \sigma_{max1}) - \log (X_T - \sigma_{max3}) = \gamma(\log\sigma_{max1} - \log\sigma_{max3}) + \log(N_{f1}^{\beta}-1) - \log(N_{f3}^{\beta}-1) \quad (5.14)$$

From above eq. (7) and (8) we get following two value of γ as follows

$$\gamma=\log(X_T-\sigma_{max1})-\log(X_T-\sigma_{max2})-\log(N_{f1}^{\beta}-1)+\log(N_{f2}^{\beta}-1)/(\log\sigma_{max1}-\log\sigma_{max2}) \quad (5.15)$$

$$\gamma=\log(X_T-\sigma_{max1})-\log(X_T-\sigma_{max3})-\log(Nf_1^{\beta}-1)+\log(N_{f3}^{\beta}-1)/(\log\sigma_{max1}-\log\sigma_{max3}) \quad (5.16)$$

Using the above two equations (5.15) & (5.16) we can determine γ by interactive procedure, thus we get values of γ and β.

To calculate α we simply use any of the eq. (5.8) ,(5.9) or (5.10)

$$\alpha = (X_T-\sigma_{max1}) / [X_T^{\ 1-\gamma}\ \sigma_{max1}^{\gamma}(1-R)^{\gamma}\ (N_{f1}^{\beta} - 1) / f^{\beta}] \quad (5.17)$$

Thus we determine the value of α , β , γ.

Once we know the α , β , γ fatigue life N_f , residual strength X_T^r at some new stress ratio (R_1) or new frequency (f_1) can be calculated as follows-

Fatigue life :

$$N_f = \left[\frac{(X_T - \sigma_{max1})\, f_1^{\beta}}{\alpha\, X_T^{\,1-\gamma}\, \sigma_{max}^{\gamma} (1-R)^{\gamma}} + 1 \right]^{1/\beta} \tag{5.18}$$

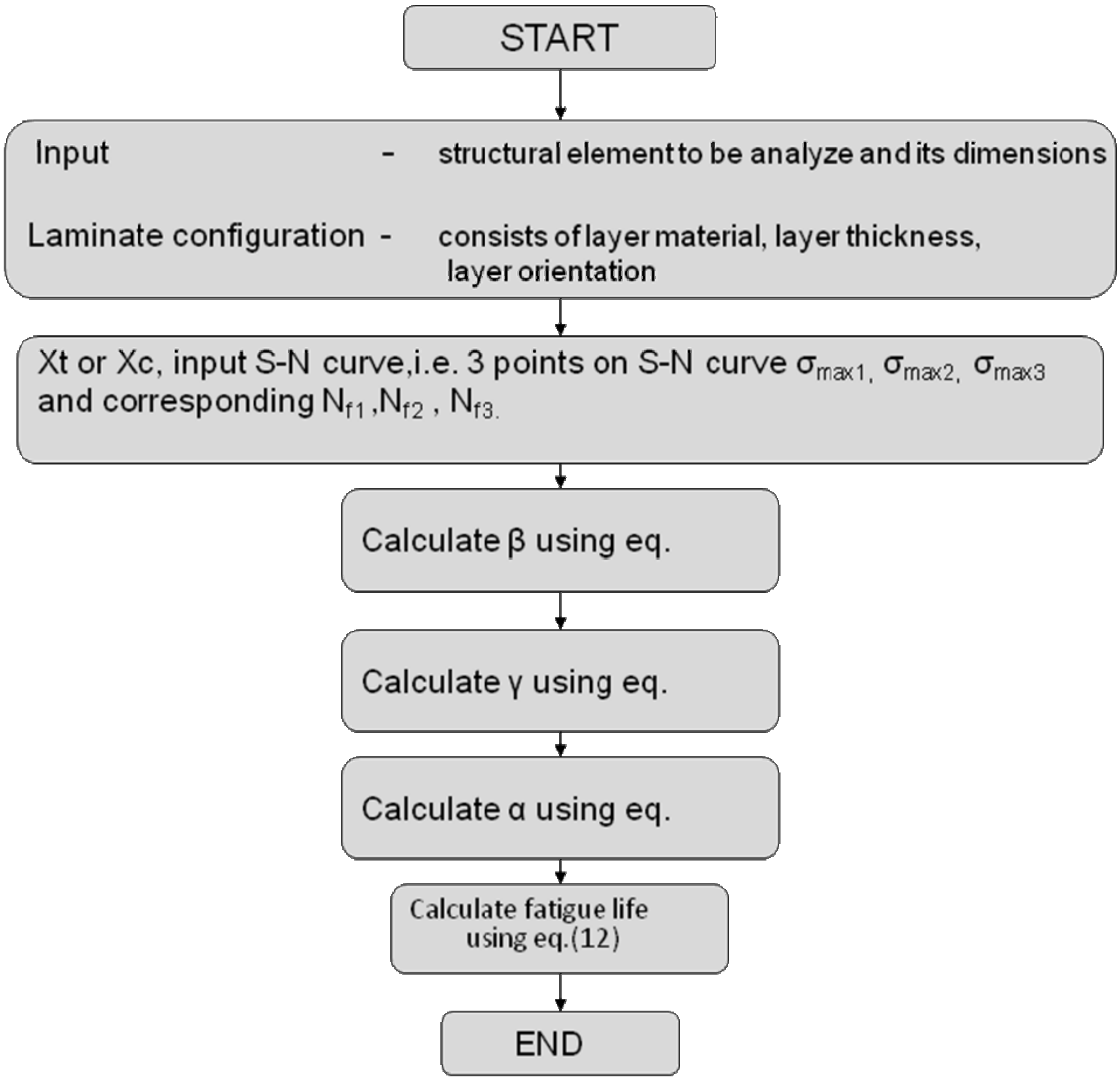
.

Residual strength:

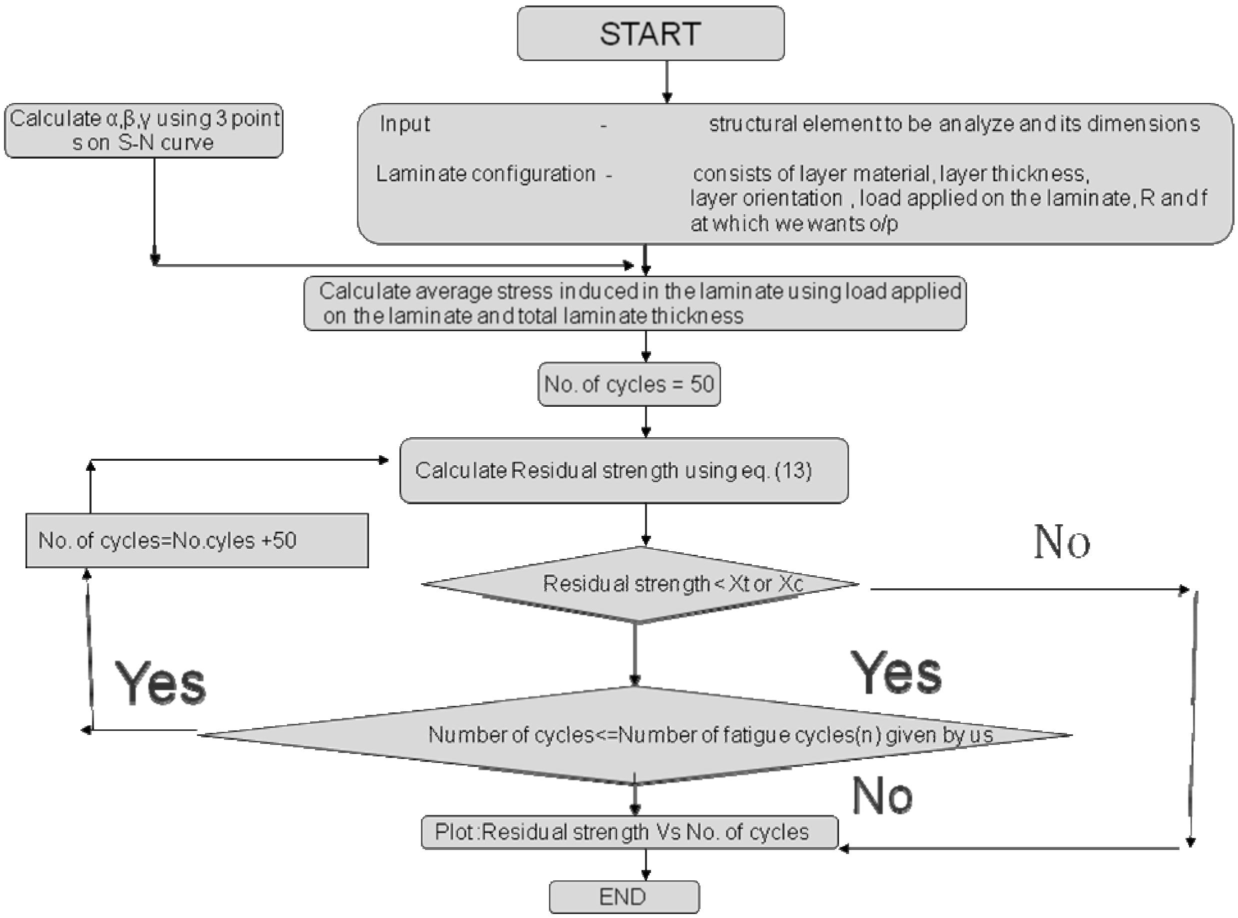
$$X_T^{r} = X_T - [\alpha\, X_T^{\,1-\gamma}\, \sigma_{max}^{\gamma} (1-R)^{\gamma} (N_f^{\beta} - 1) / f^{\beta}] \tag{5.19}$$

5.4Flow charts

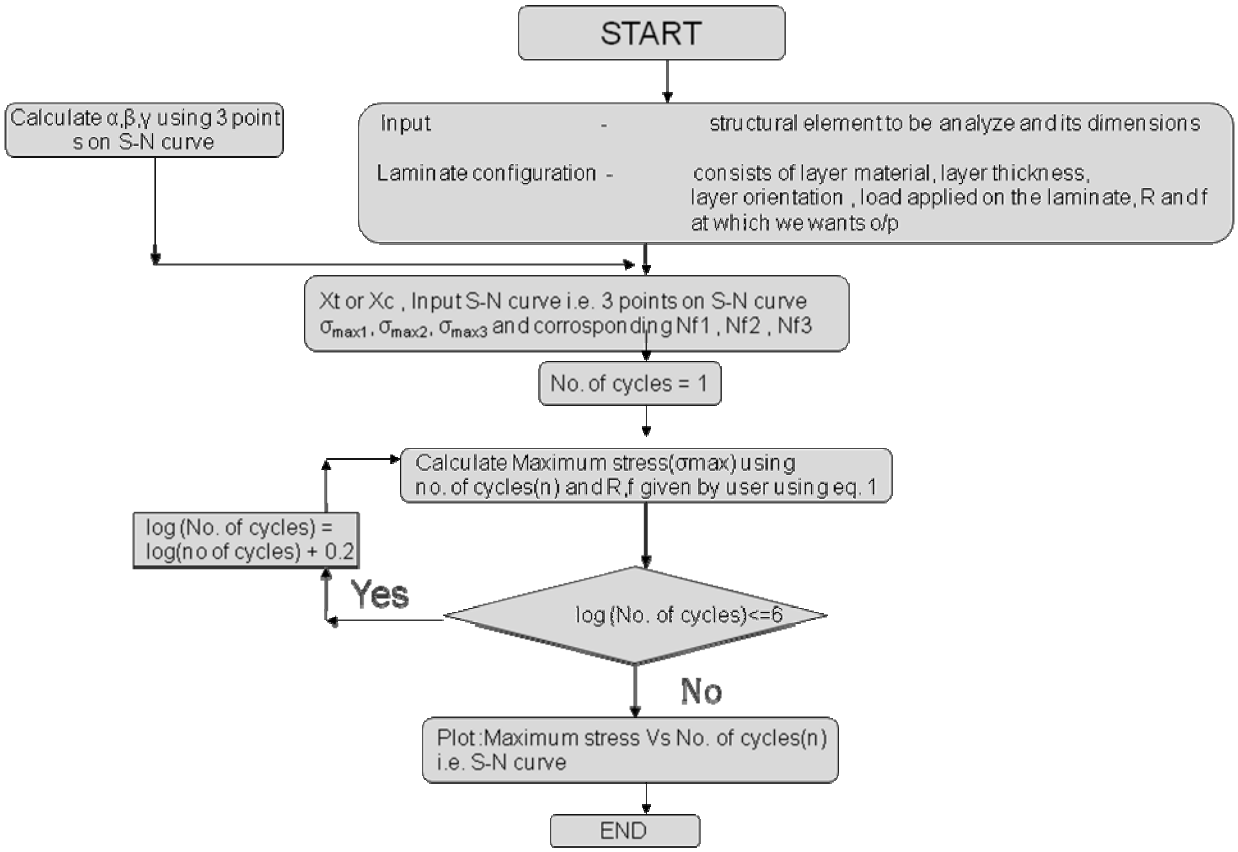
5.4.1 To Determination of "Fatigue life" of Composites



5.3.2 To Plot "Residual strength Vs Number of cycles"



5.4.3 To Plot "Maximum stress Vs Fatigue Life"


START
Calculate α,β,γ using 3 point s on S-N curve
Input - structural element to be analyze and its dimensions
Laminate configuration - consists of layer material, layer thickness, layer orientation , load applied on the laminate, R and f at which we wants o/p
Xt or Xc , Input S-N curve i.e. 3 points on S-N curve σmax1, σmax2, σmax3 and corrosponding Nf1 , Nf2 , Nf3
No. of cycles = 1
Calculate Maximum stress(σmax) using no. of cycles(n) and R,f given by user using eq. 1
log (No. of cycles) = log(no of cycles) + 0.2
Yes
log (No. of cycles)<=6
No
Plot :Maximum stress Vs No. of cycles(n) i.e. S-N curve
END

CHAPTER 6

RESULTS

6.1 Results of Predicted Properties for Lamina

Detailed discussion of mathematical formulation [Jacob Aboudi and Marek-Jerzy Pindera (19)] for all uniaxial lamina properties subjected to in-plane loading for inverse micromechanics has been presented in chapter 3. The comparisons of actual and by "Method of Cells" are as in references.

6.1.1 Input "fiber" Properties for Orthotropic Lamina Calculations

Table 6.1 Fiber Properties

INPUT DATA		
fiber volume fraction	0.6	
Fiber =>	**Graphite**	
n =>	**1 (default value)**	
Properties		**unit**
Young's modulus of fiber in longitudinal direction (Elf)	233000	MPa
Young's modulus of fiber in transverse direction (Etf)	23100	MPa
Shear modulus of fiber in longitudinal direction (Gltf)	8960	MPa
Poisson's ratio of fiber in longitudinal direction (nltf)	0.2	–
Poisson's ratio of fiber in transverse direction (nttf)	0.4	–
Tensile strength of fiber (Xtf)	2250	MPa
Compressive strength of fiber (Xcf)	2000	MPa
Longitudinal thermal conductivity of fiber (klf)	12	W/m^0K
Transverse thermal conductivity of fiber (ktf)	15	W/m^0K
Longitudinal coefficient of thermal expansion (αlf)	-0.54	$10^{-6}/{}^0K$
Transverse coefficient of thermal expansion (αtf)	10.10	$10^{-6}/{}^0K$
Density of fiber (ρf)	1800	kg/m^3
Coefficient of moisture expansion of fiber (blf)	NA	10^{-6}/%M
Coefficient of moisture expansion of fiber (btf)	NA	10^{-6}/%M

6.1.2 Input "matrix" Properties For Orthotropic Lamina Calculations

Table 6.2 Matrix Properties

Matrix =>	**Polymer**	
Properties		**unit**
Young's modulus of matrix in longitudinal direction (Elm)	4620	MPa
Poisson's ratio of matrix in longitudinal direction (nltm)	0.36	–
Tensile strength of matrix (Xtm)	60	MPa
Compressive strength of matrix (Xcm)	200	MPa
Shear strength of matrix (Xcm)	110	MPa
Longitudinal thermal conductivity of matrix (klm)	0.19	$W/m^{0}K$
Transverse thermal conductivity of matrix (ktm)	0.19	$W/m^{0}K$
Longitudinal coefficient of thermal expansion (αlm)	4.14E+01	$10^{-6}/{}^{0}K$
Density of matrix (ρm)	1200	kg/m^3
Coefficient of moisture expansion of matrix (bm)	0.14	$10^{-6}/\%M$

6.1.3 Calculated Properties of Orthotropic "lamina"

Table 6.3 Calculated Properties Of Lamina

OUTPUT DATA			
Fiber =>	**Graphite**		
Matrix =>	**Polymer**		
Lamina =>	**Graphite/Polymer**	**n =>**	**1**
Fiber volume fraction =>		**0.6**	
Properties			**unit**
ELASTIC PROPERTIES OF LAMINA			
Axial Young's modulus of lamina (E_1)		141678	MPa
Transverse Young's modulus of lamina (E_2)		11152.9	MPa
Transverse Young's modulus of lamina (E_3)		11152.9	MPa
Poisson's ratio of lamina in 1-2 plane (n_{12})		0.26	MPa
Poisson's ratio of lamina in 1-3 plane (n_{13})		0.26	MPa
Poisson's ratio of lamina in 2-3 plane (n_{23})		0.47	MPa
Axial Shear Modulus of lamina (G_{12})		3919.5	MPa
Transverse Shear Modulus of lamina (G_{13})		3919.5	MPa
Transverse Shear Modulus of lamina (G_{23})		3790.01	MPa
STRENGTHS OF LAMINA			
Tensile strength of lamina along direction 1 or x (X_t)		1368.5	MPa
Tensile strength of lamina along direction 2 or y (Y_t)		51.473	MPa
Tensile strength of lamina along direction 3 or z (Z_t)		51.473	MPa
Compressive strength of lamina along direction 1 or x (X_c)		1216.45	MPa
Compressive strength of lamina along direction 2 or y (Y_c)		171.575	MPa
Compressive strength of lamina along direction 3 or z (Z_c)		171.575	MPa
Shear strength of lamina in 1-2 plane (S_{12})		94.438	MPa
Shear strength of lamina in 1-3 plane (S_{13})		94.438	MPa
Shear strength of lamina in 2-3 plane (S_{23})		64.25	MPa
THERMAL PROPERTIES OF LAMINA			
Thermal conductivity of lamina in direction 1 (k_1)		18	W/m^0K
Thermal conductivity of lamina in direction 2 (k_2)		19.86	W/m^0K
Thermal conductivity of lamina in direction 3 (k_3)		19.86	W/m^0K
Coefficient of thermal expansion in direction 1 (α_1)		0.0686	$10^{-6}/ {}^0K$
Coefficient of thermal expansion in direction 2 (α_2)		26.96	$10^{-6}/ {}^0K$
Coefficient of thermal expansion in direction 3 (α_3)		26.96	$10^{-6}/ {}^0K$
MOISTURE PROPERTIES OF LAMINA			
Coefficient of moisture expansion of lamina in drec. 1 (b_1)		0.0059	10^{-6}/%M
Coefficient of moisture expansion of lamina in drec. 2 (b_2)		0.246	10^{-6}/%M
Coefficient of moisture expansion of lamina in drec. 3 (b_3)		0.246	10^{-6}/%M
Density of lamina (ρ)		1560	kg/m^3

6.2 Results of Vibration Analysis

Detailed discussion and mathematical modal [Arthur W. Leissa & Yoshihiro Narita(22)] for free-vibration frequencies of symmetrically laminated, simply supported plates, cantilever plates ,plates having all edges free presented in chapter4. T
he Rayleigh-Ritz method with algebraic polynomial displacement functions is used to solve the vibration problem.

6.2.1 Input Material Properties for Composite Plates

Mohamad S. Qatu[26]

Table6.4

Composite material	E-glass/epoxy (E/E)
Axial Young's modulus of lamina (E_1)	98 Gpa
Transverse Young's modulus of lamina (E_2)	7.9 Gpa
Axial Shear Modulus of lamina (G_{12})	5.16 Gpa
Poisson's ratio (ν)	0.28
Dimensions	a=304.8 mm, b=76.2 mm,
h/ply	0.134 mm
Density of lamina (ρ)	1520 kg/m3

6.2.2 Natural Frequencies for Cantilever Laminated Plates

By Rayleigh-Ritz method (successive approximations approach to solve)

	Rayleigh-Ritz method	Experimental Results
Mode 1	11.1	11.23
Mode 2	39.5	42.47
Mode 3	69.4	70.58
Mode 4	129.11	130.62
Mode 5	193.93	197.49
Mode 6	261.21	262.13
Mode 7	379.92	396.21
Mode 8	396.22	401.37

6.2.3 Natural Frequencies For Simply Supported Symmetrically Laminated Rectangular Plates

By Rayleigh-Ritz method (successive approximations approach to solve)

	Rayleigh-Ritz method	Experimental Results
Mode 1	138.85	139.36
Mode 2	341.25	343.67
Mode 3	403.56	406.59
Mode 4	475.74	479.23
Mode 5	631.19	635.78
Mode 6	705.54	711.33
Mode 7	799.28	806.23
Mode 8	887.93	896.96

6.2.4 Natural Frequencies for laminated rectangular plates having all edges free

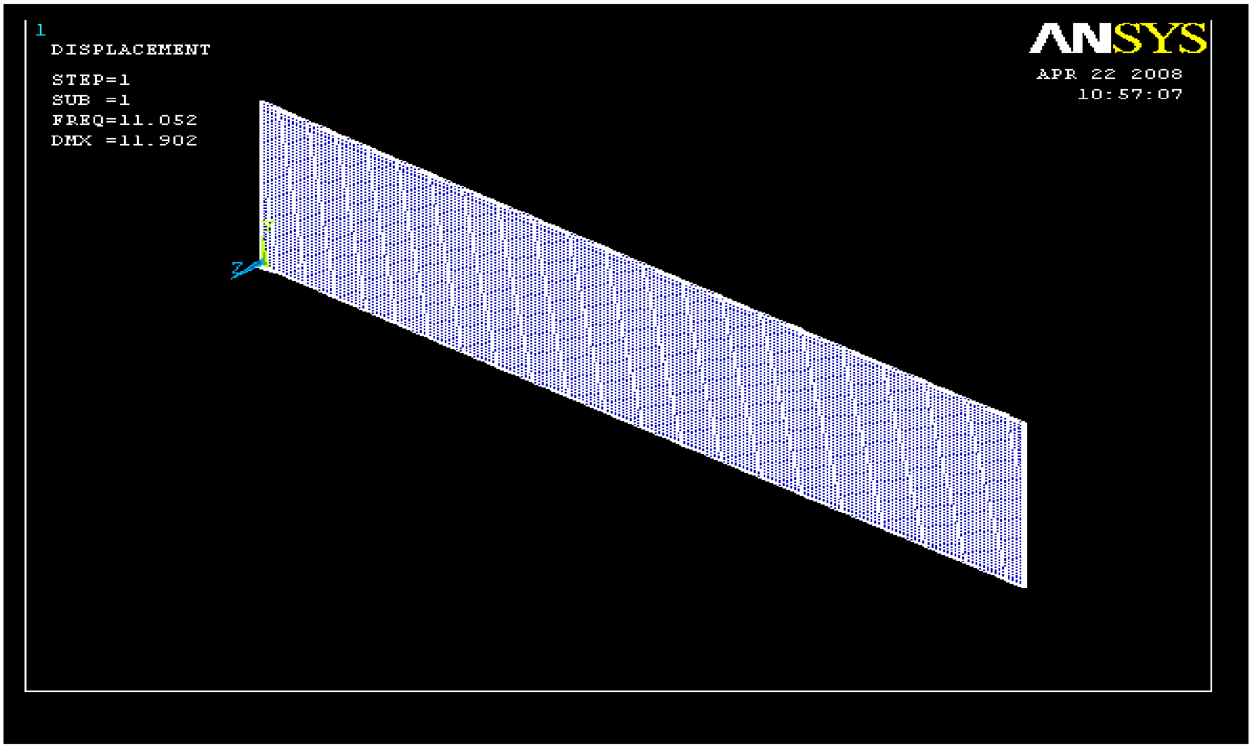
By Rayleigh-Ritz method (successive approximations approach to solve)

	Rayleigh-Ritz method	Experimental Results
Mode 1	60.12	61.78
Mode 2	66.29	68.91
Mode 3	151.29	555.78
Mode 4	167.78	171.90
Mode 5	269.23	271.79
Mode 6	332.61	335.89
Mode 7	433.91	438.78
Mode 8	549.72	551.17

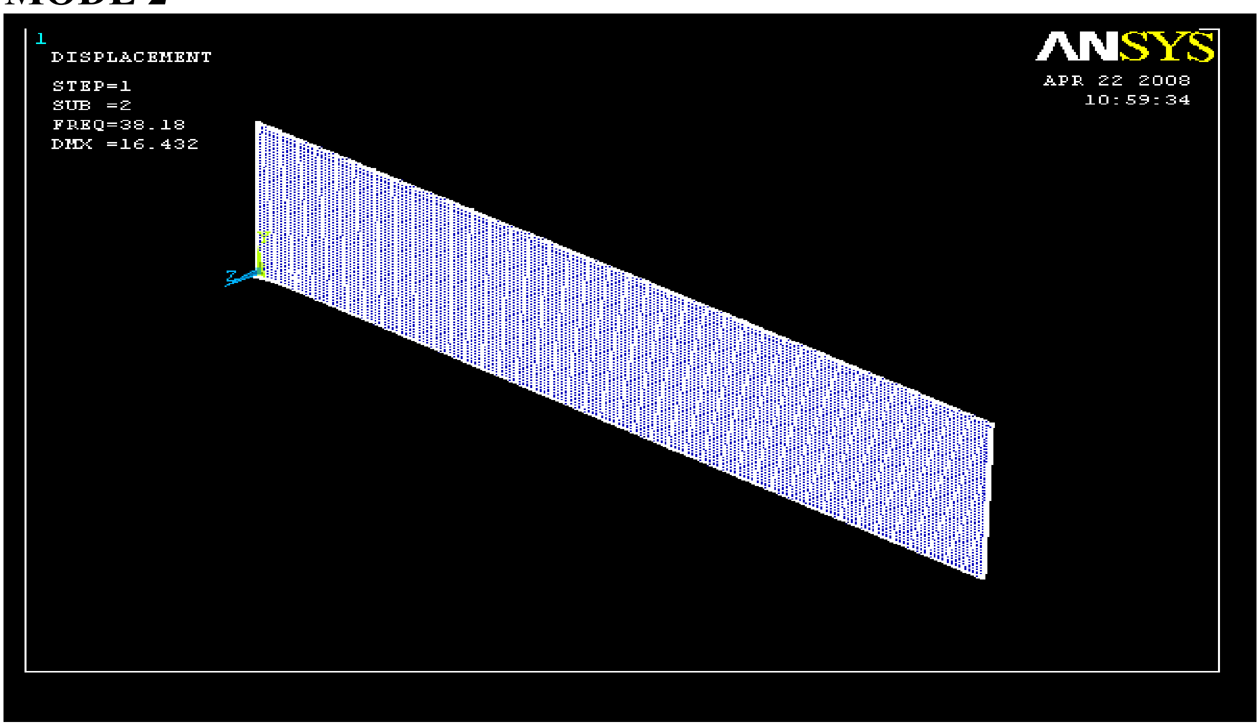
6.2.5 Comparison the Results With ANSYS 10

For cantilever laminated Plates

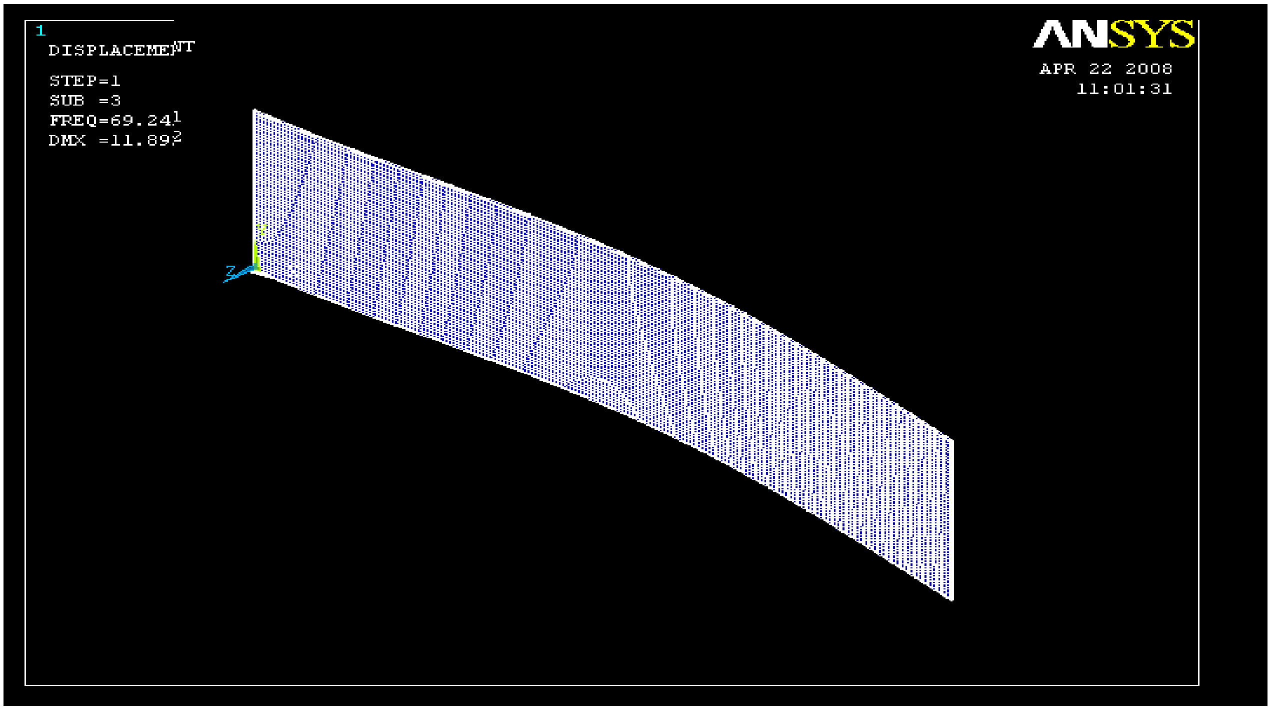
MODE 1



MODE 2



MODE3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DISPLACEMENT
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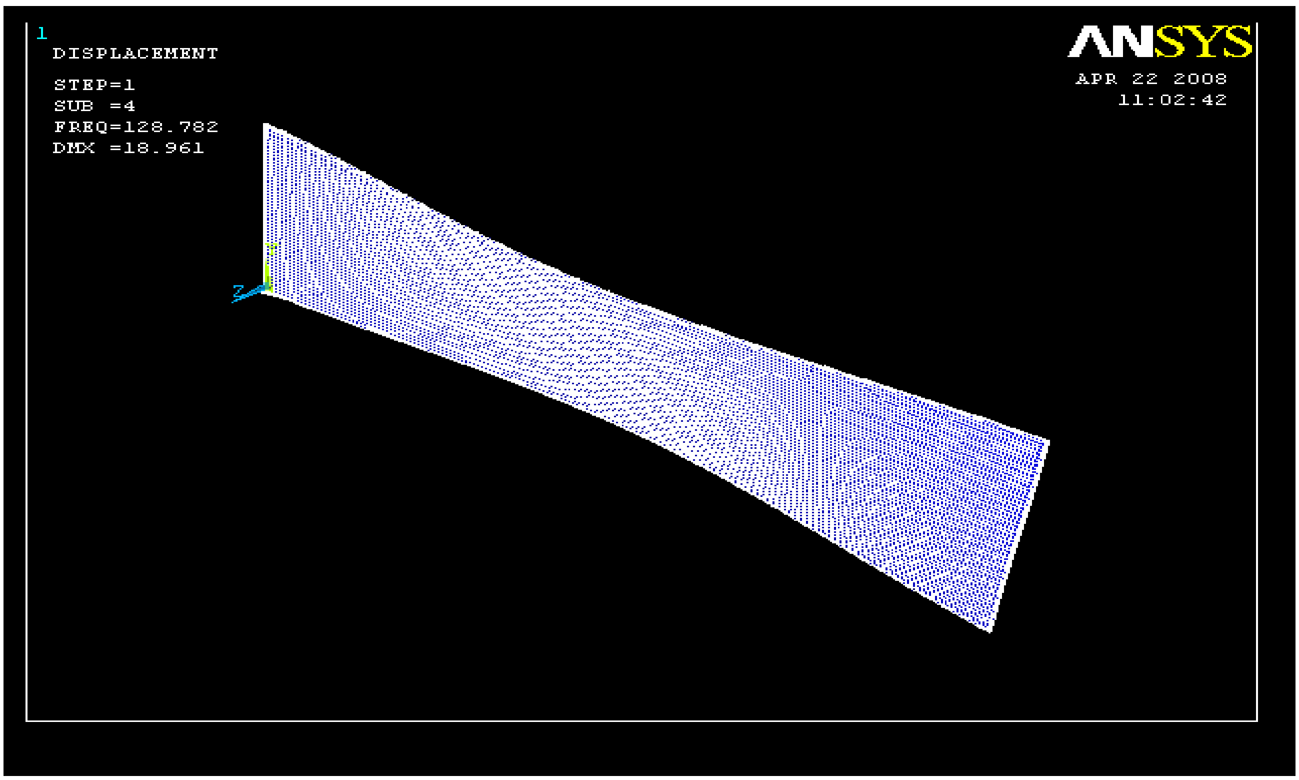
SUB =3
FREQ=69.241
DMX =11.892
ANSYS
APR 22 2008
11:01:31

MODE4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DISPLACEMENT
STEP=1
SUB =4
FREQ=128.782
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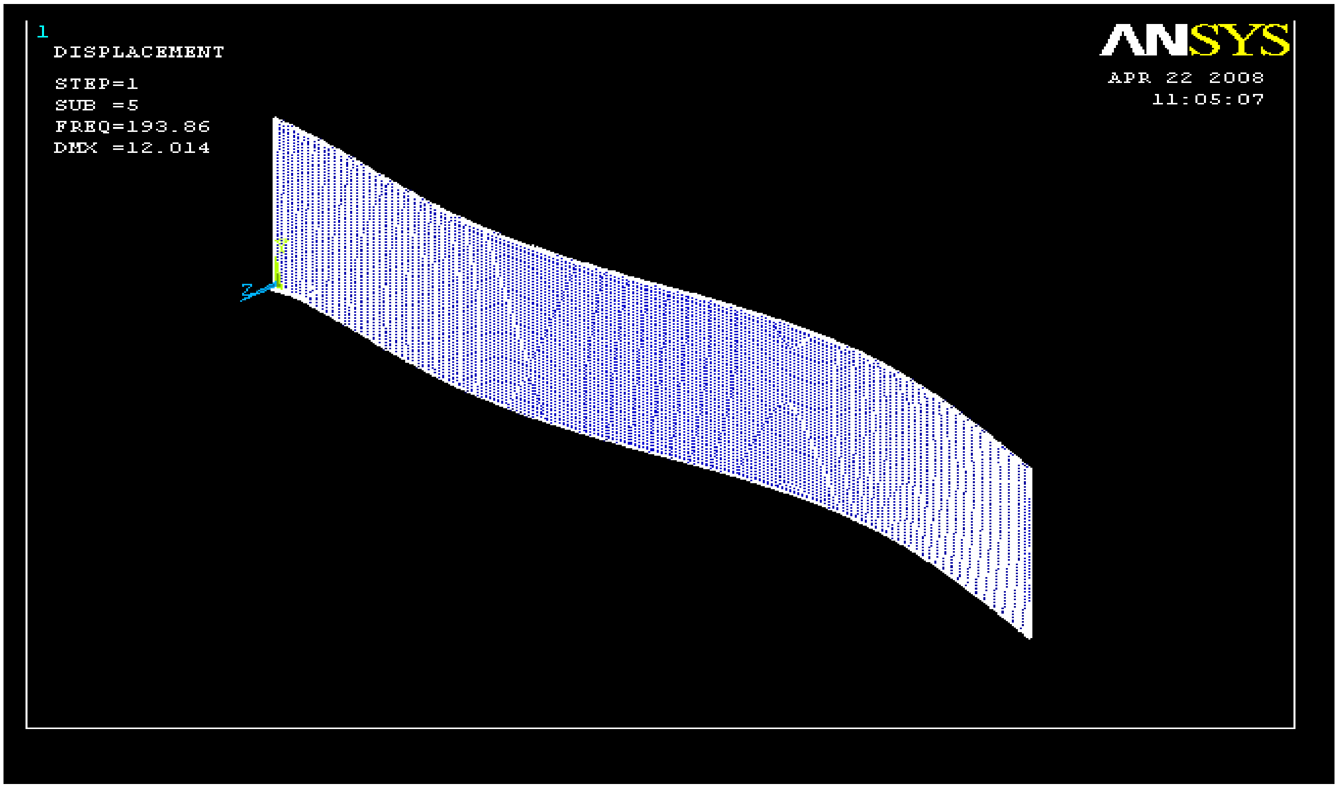
ANSYS
APR 22 2008
11:02:42

MODE5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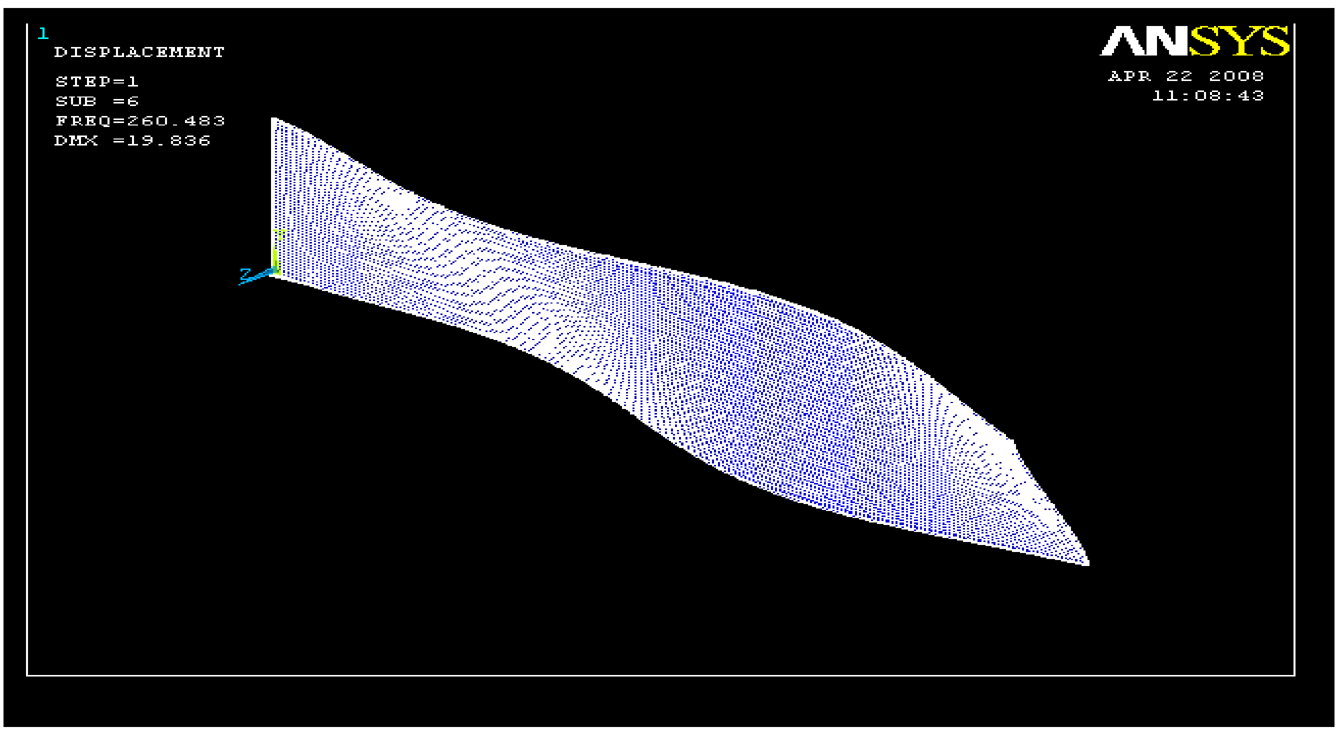
ANSYS
APR 22 2008
11:05:07

MODE6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DISPLACEMENT
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SUB =6
FREQ=260.483
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ANSYS
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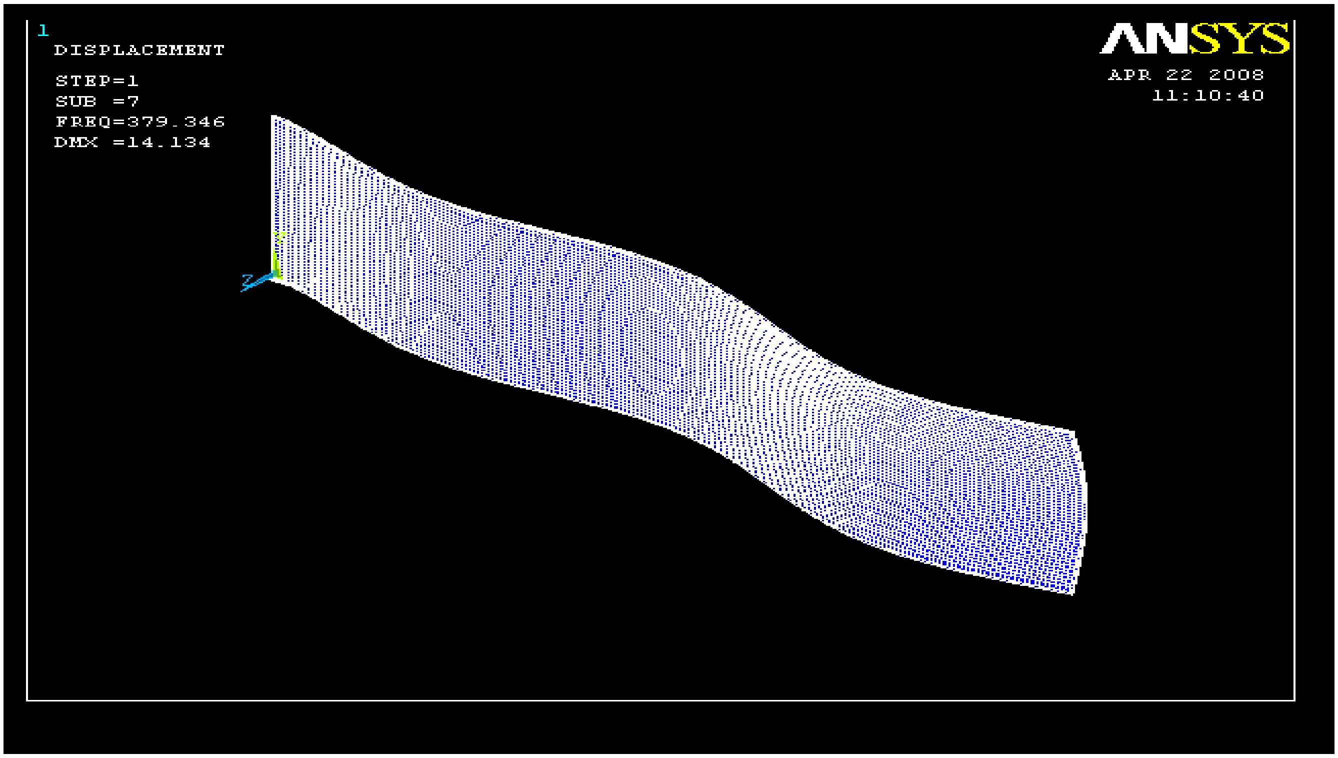
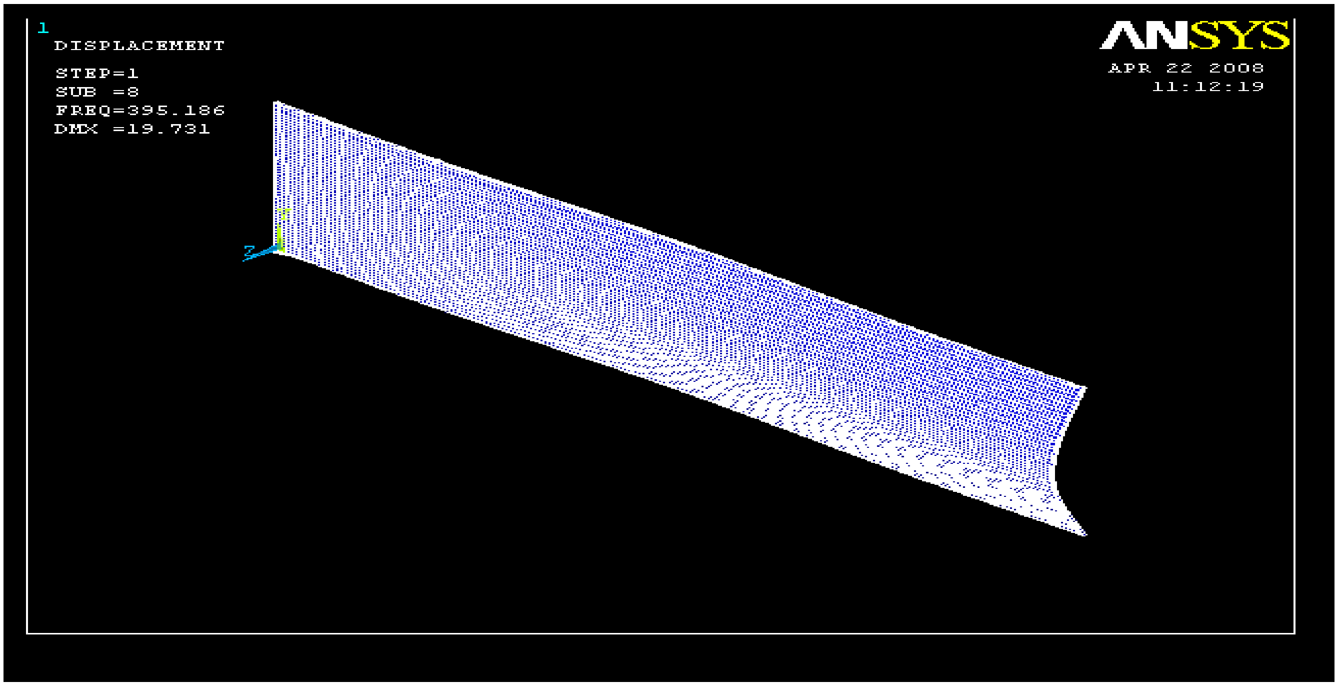
11:08:43

MODE7

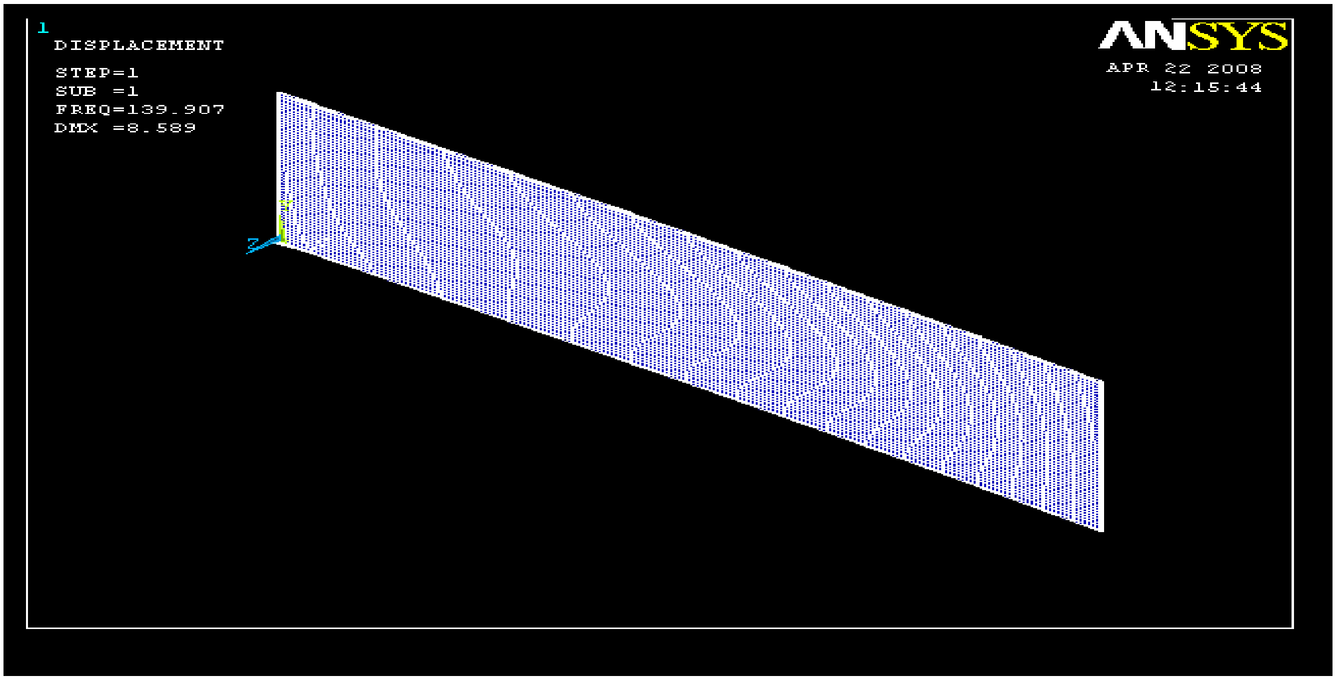

1
DISPLACEMENT
STEP=1
SUB =7
FREQ=379.346
DMX =14.134
ANSYS
APR 22 2008
11:10:40

MODE8

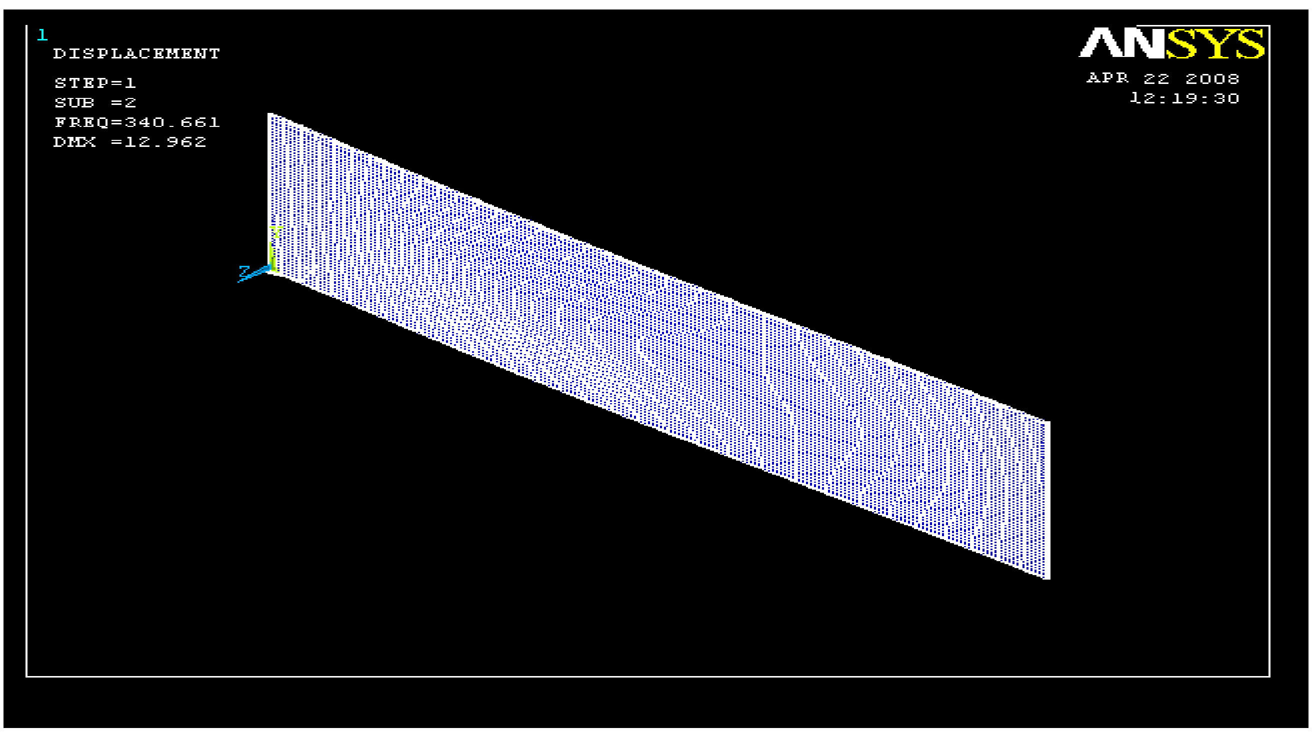

1
DISPLACEMENT
STEP=1
SUB =8
FREQ=395.186
DMX =19.731
ANSYS
APR 22 2008
11:12:19

For Simply Supported Symmetrically Laminated Rectangular Plates

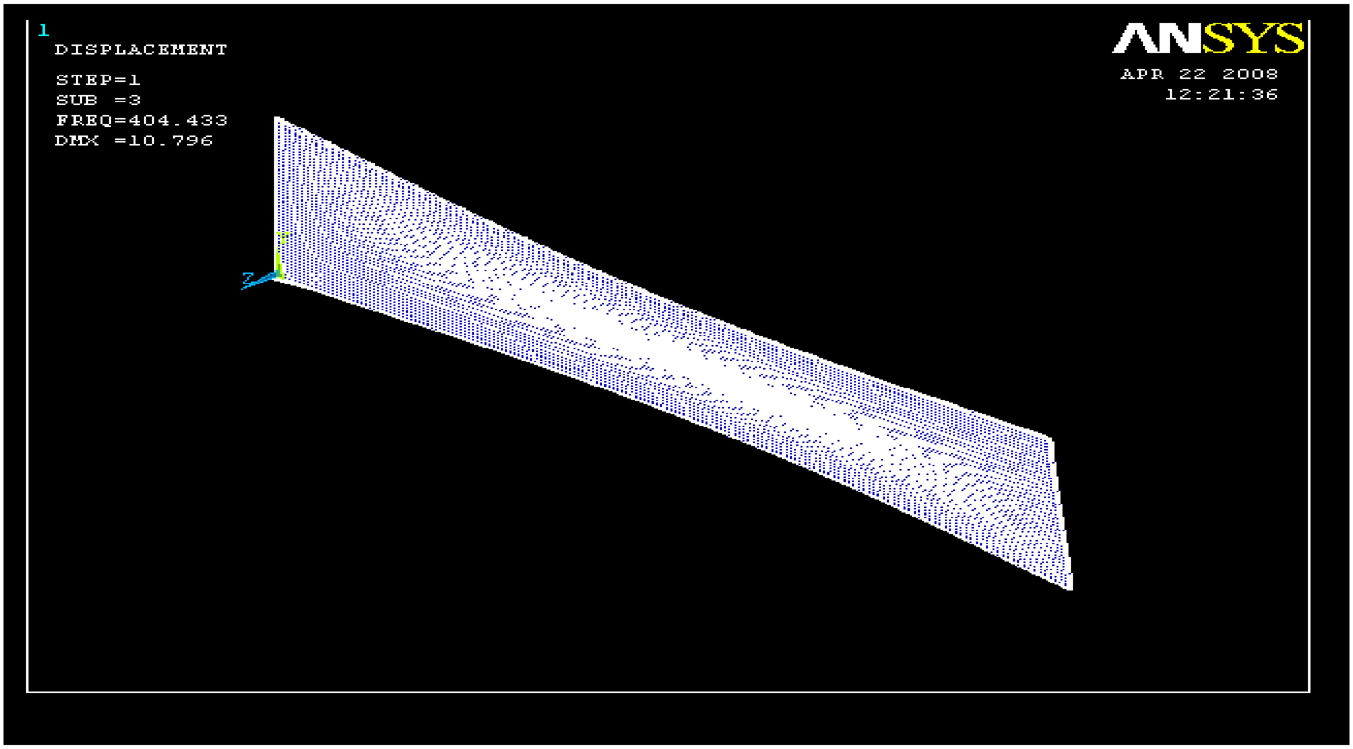
MODE 1



MODE 2



MODE 3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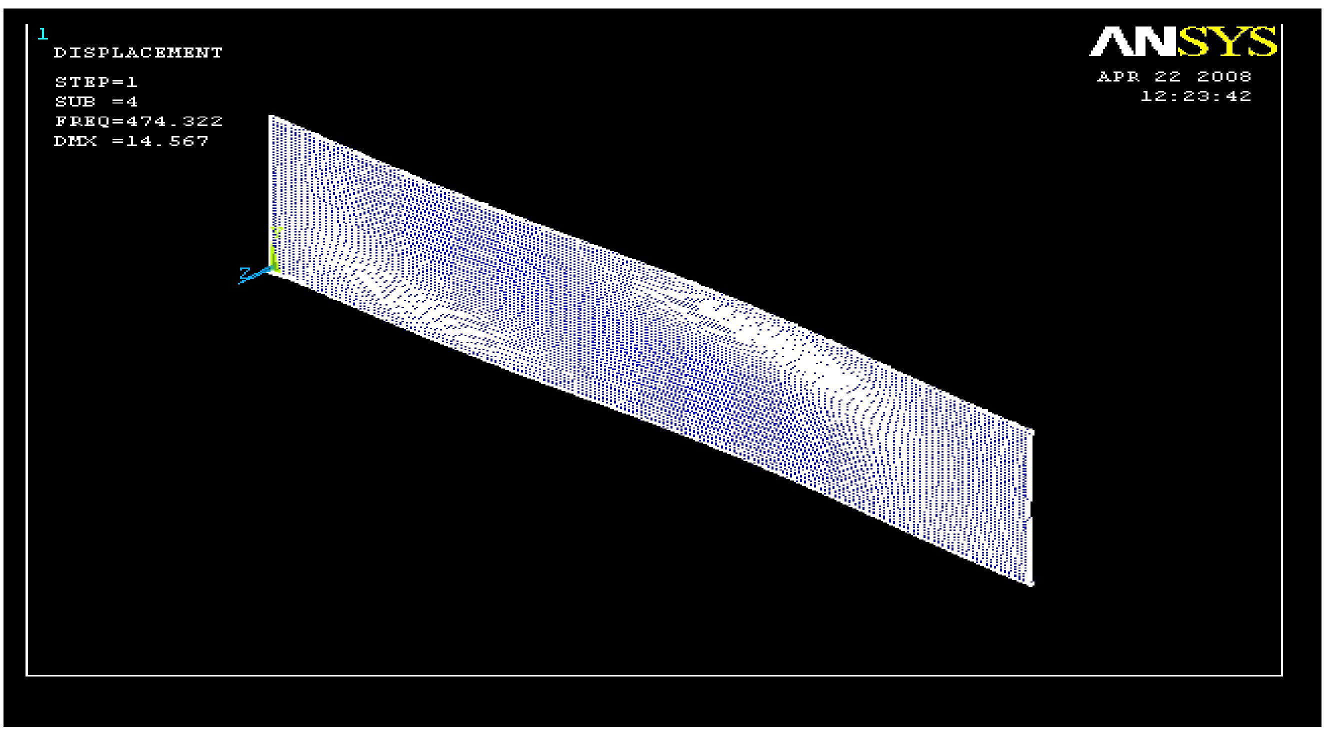
SUB =3
FREQ=404.433
DMX =10.796
ANSYS
APR 22 2008
12:21:36

MODE 4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DISPLACEMENT
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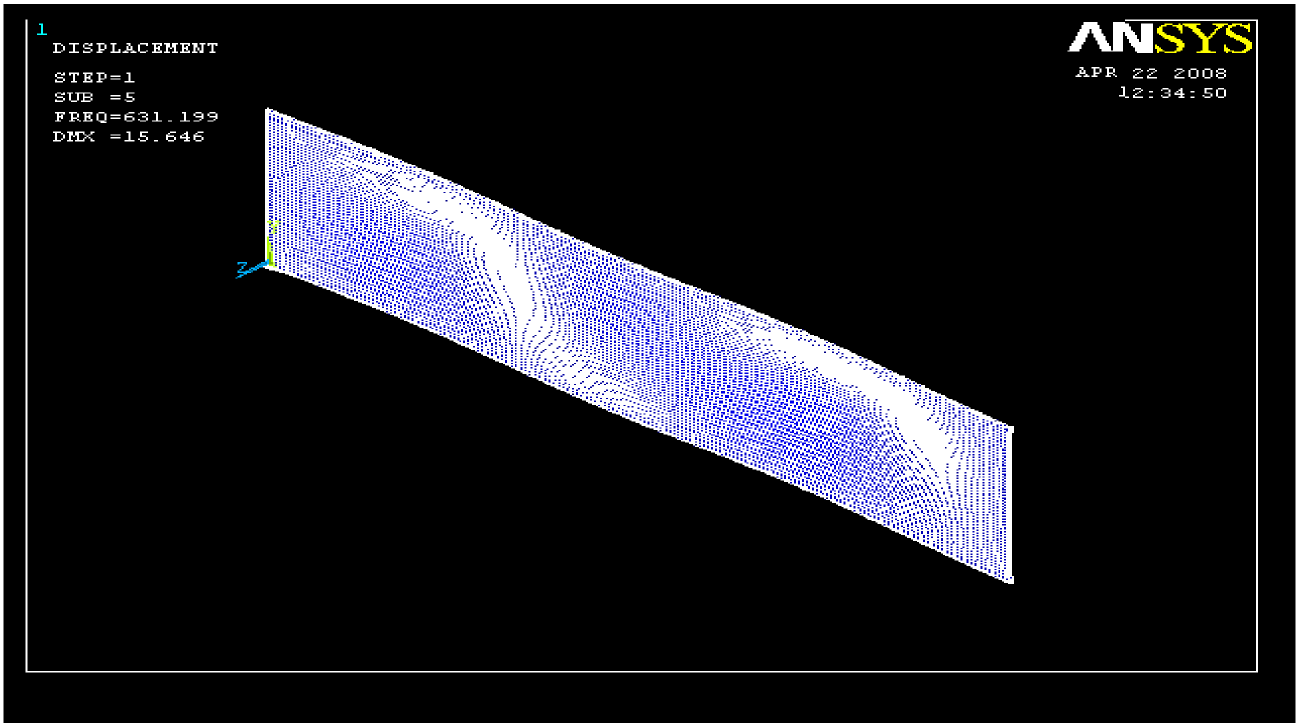
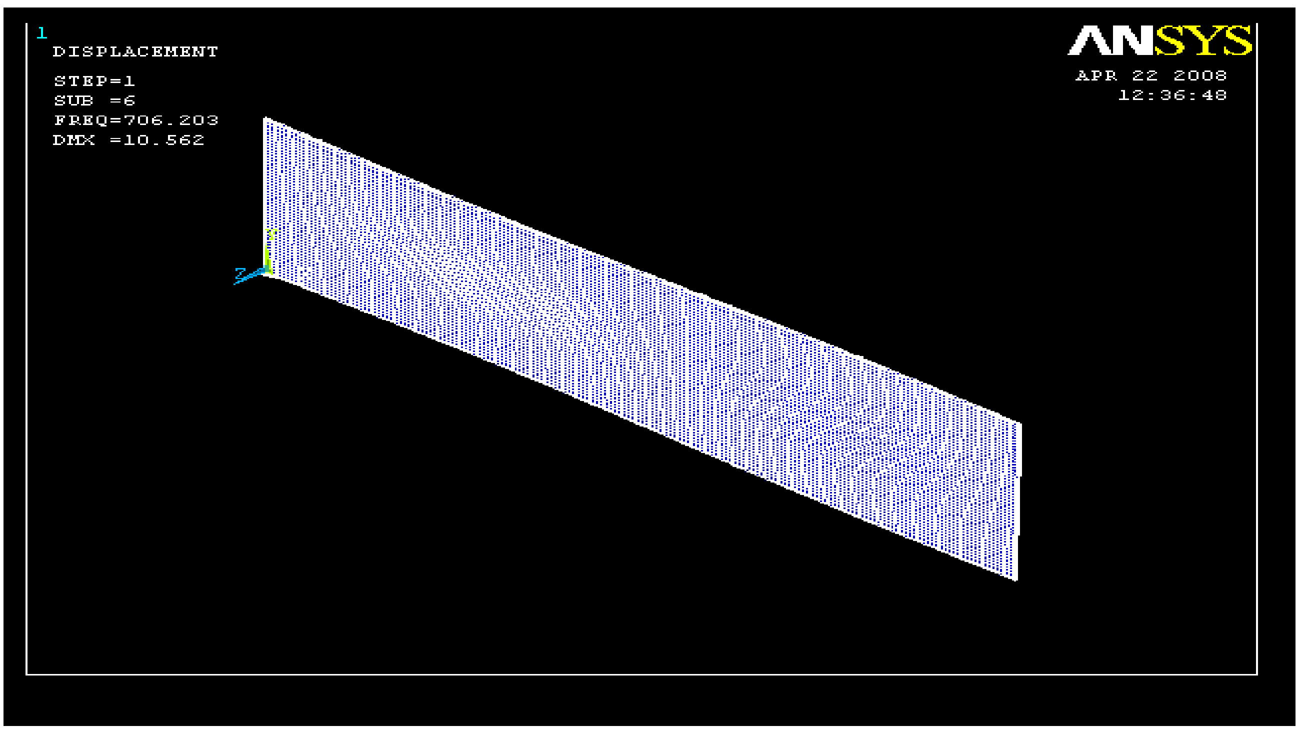
SUB =4
FREQ=474.322
DMX =14.567
ANSYS
APR 22 2008
12:23:42

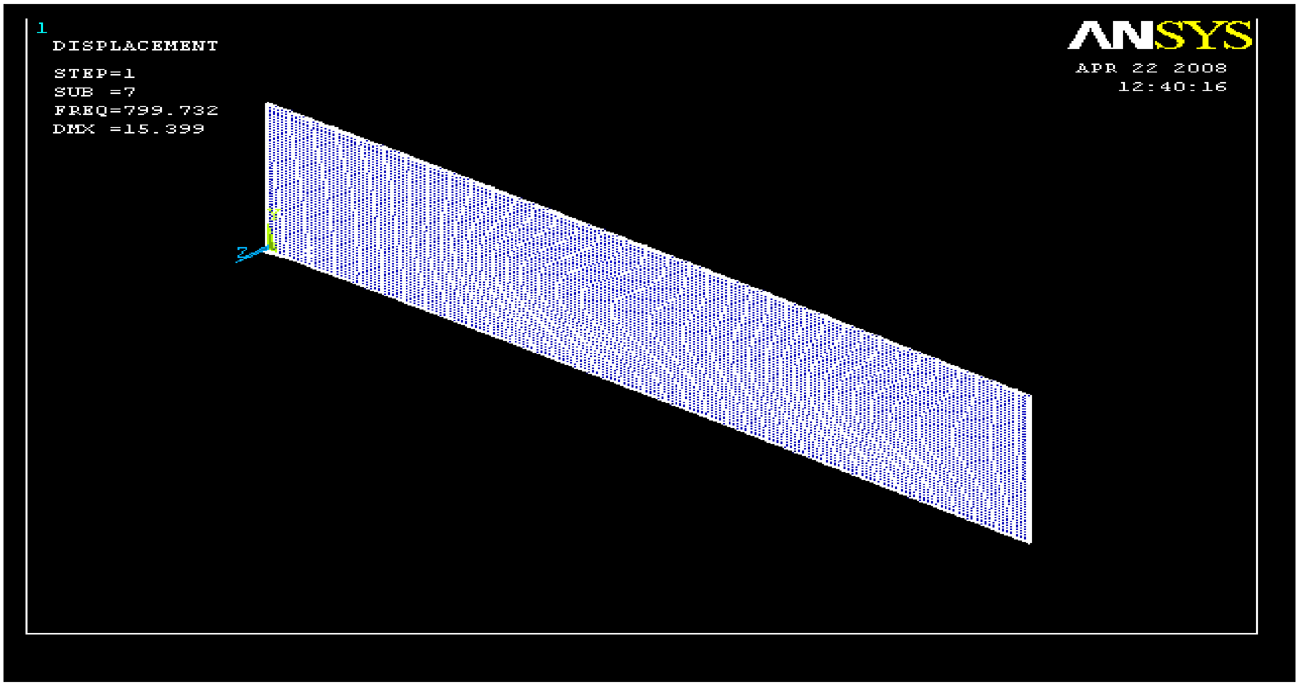
MODE 5



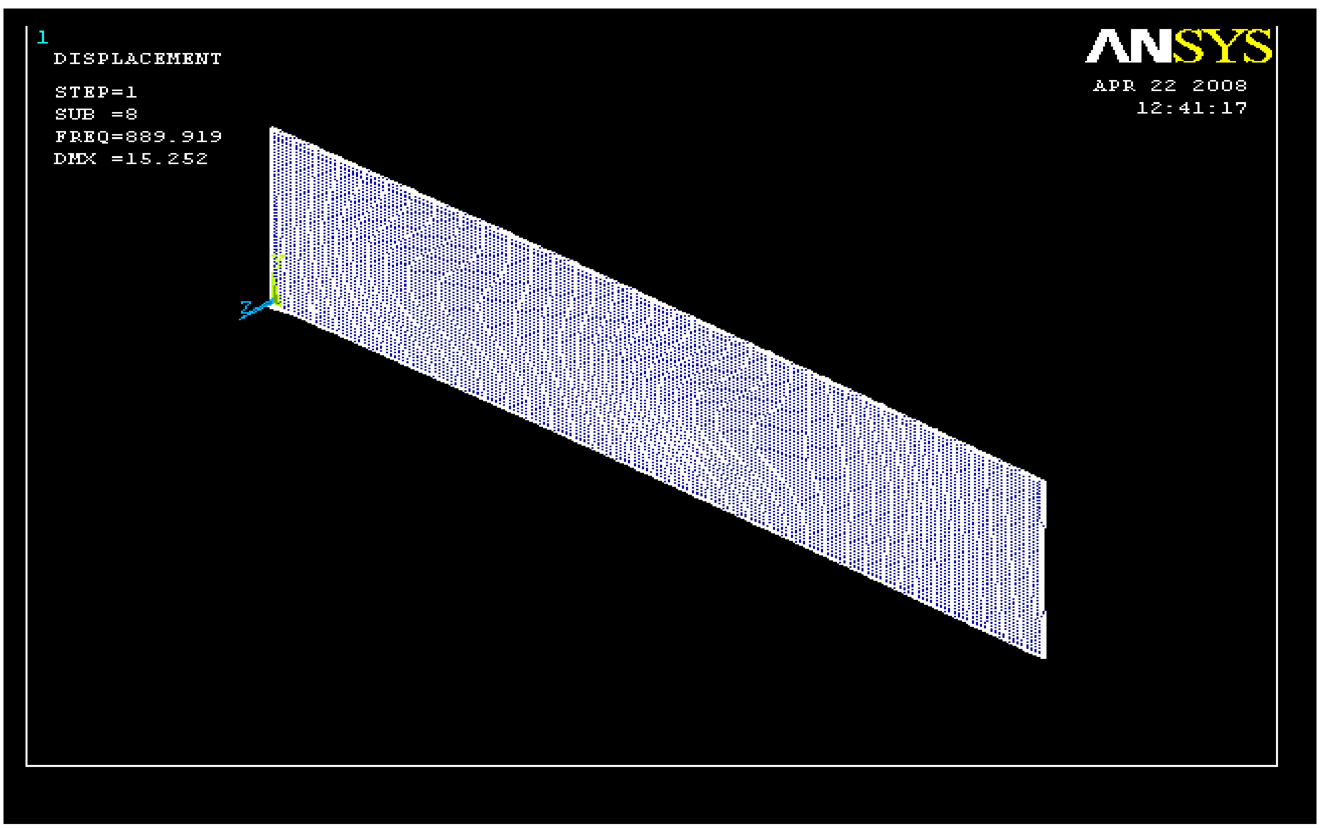
MODE 6



MODE 7

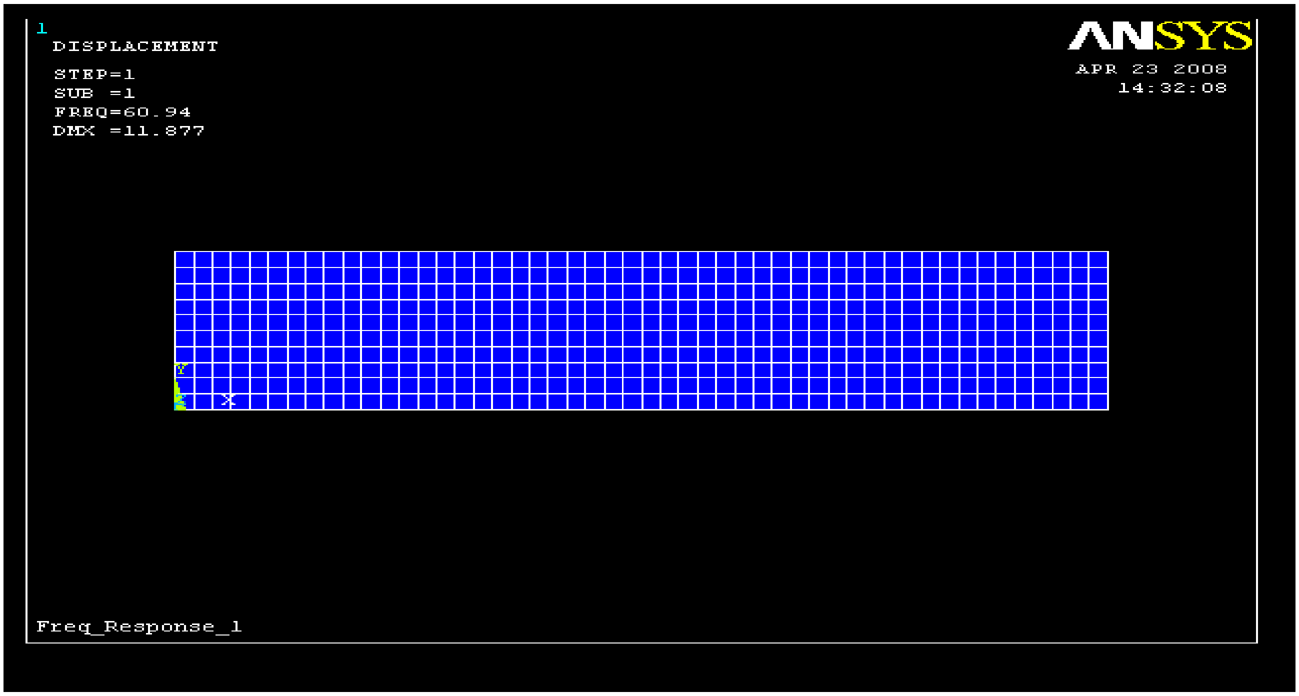


MODE 8

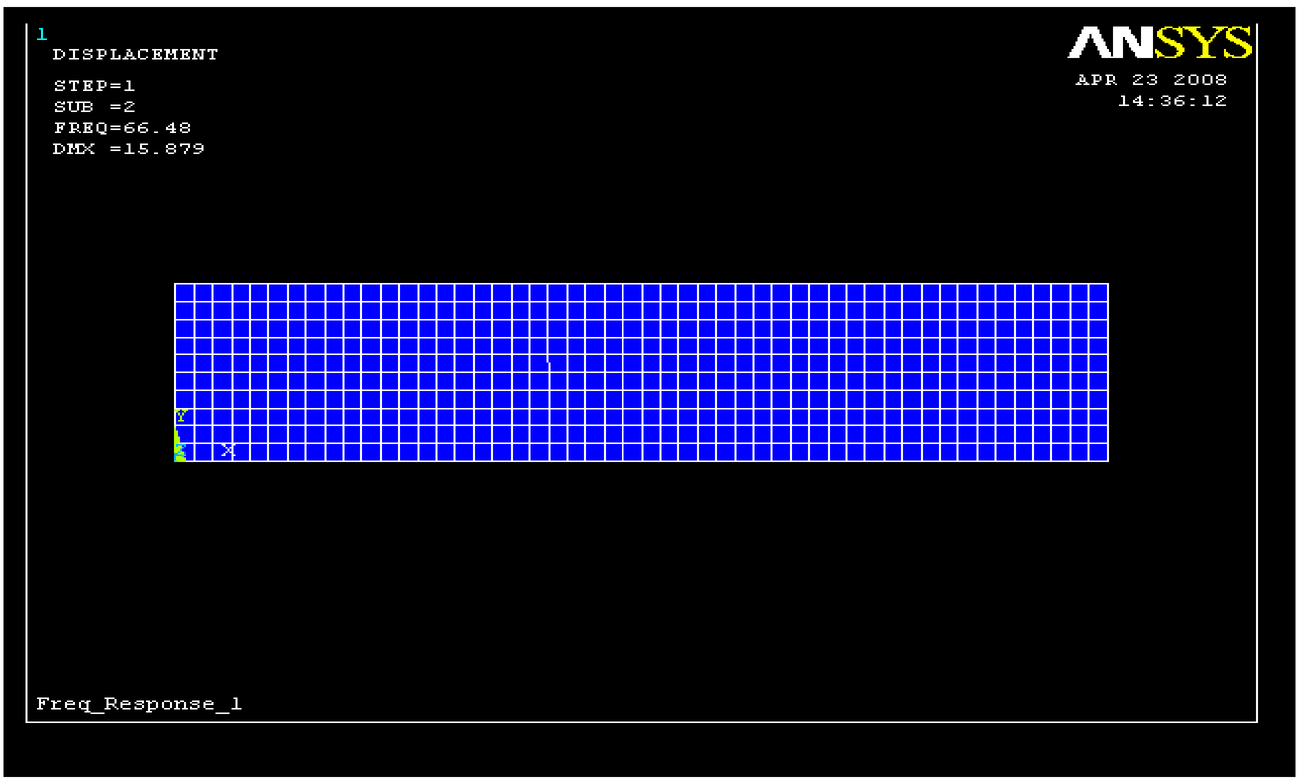


Natural Frequencies For Laminated Rectangular Plates Having All Edges Free

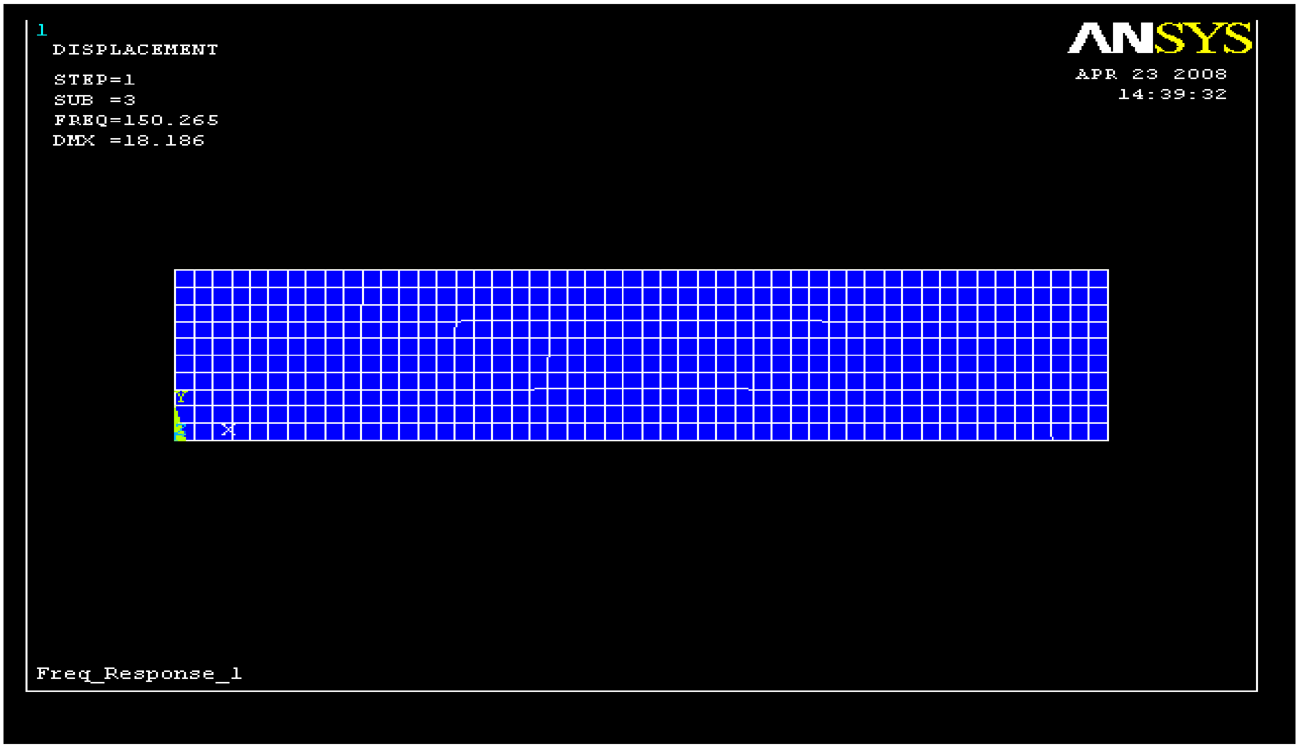
MODE 1



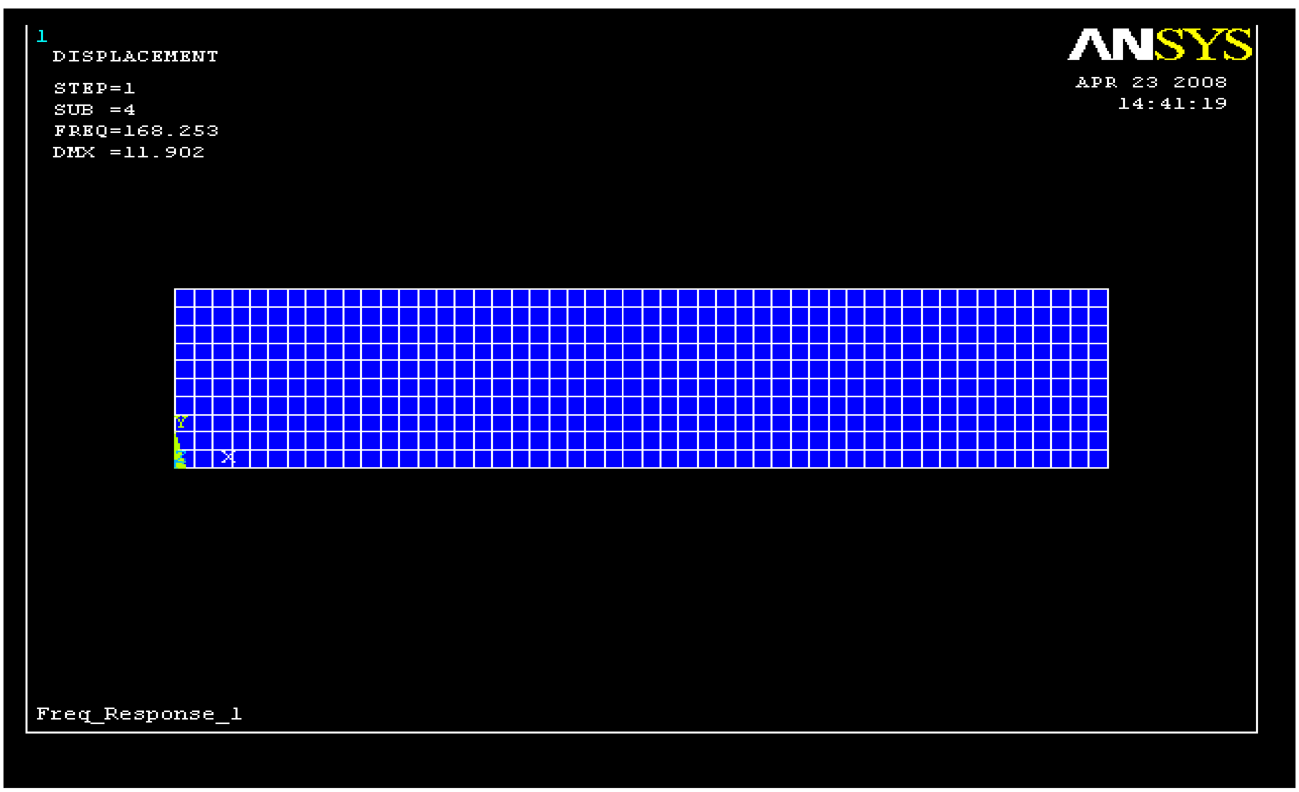
MODE 2



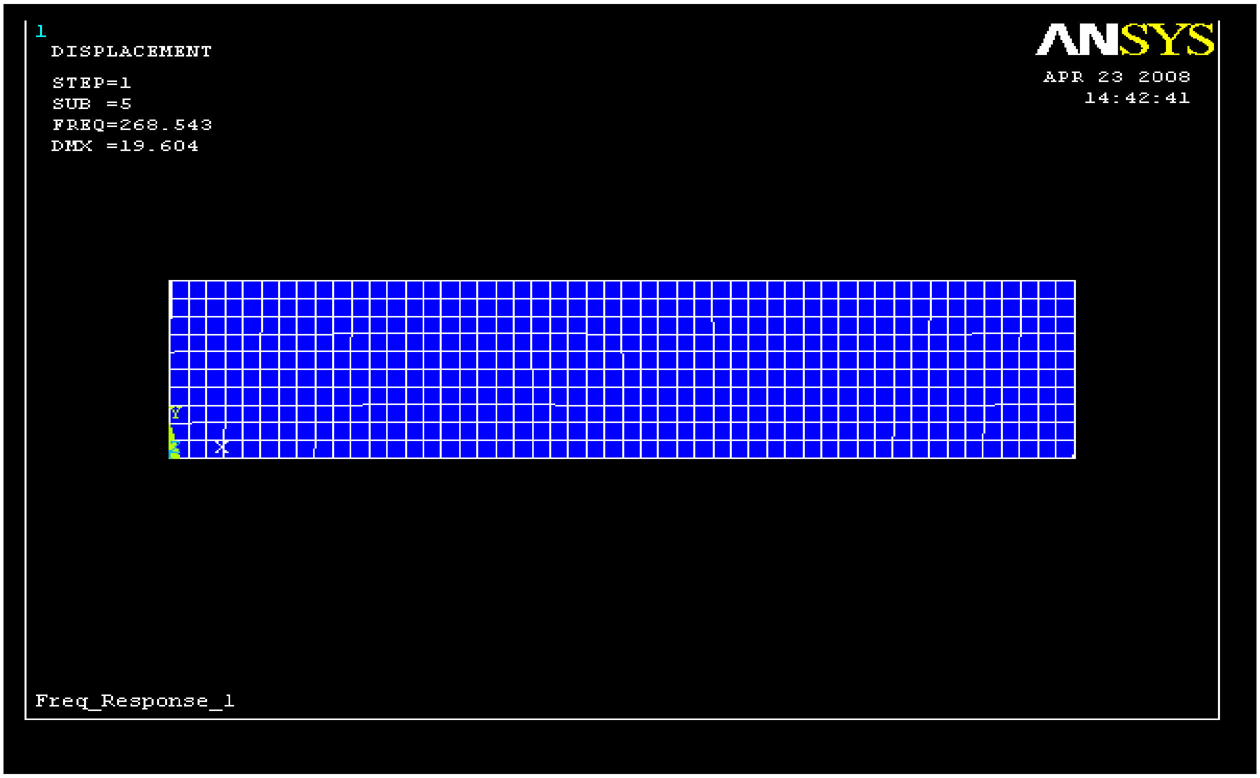
MODE3


1
DISPLACEMENT
STEP=1
SUB =3
FREQ=150.265
DMX =18.186
ANSYS
APR 23 2008
14:39:32
Y
X
Freq_Response_1

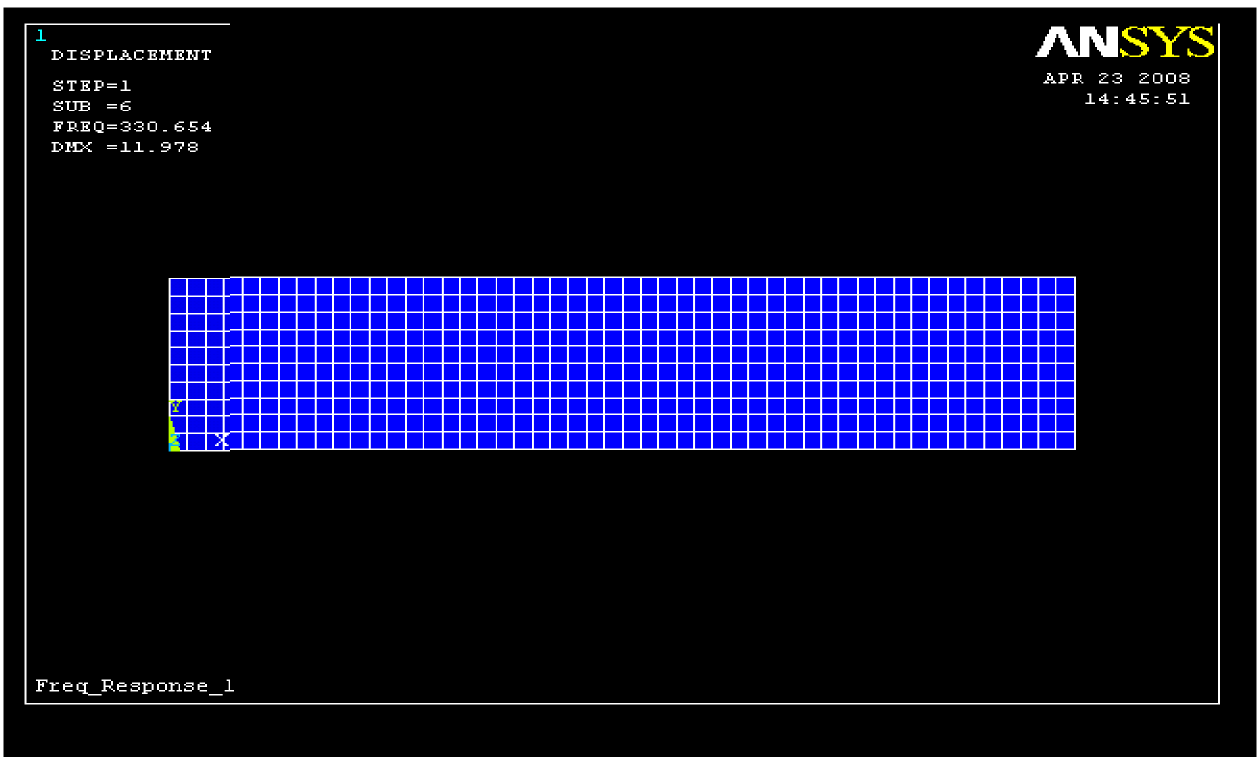
MODE4


1
DISPLACEMENT
STEP=1
SUB =4
FREQ=168.253
DMX =11.902
ANSYS
APR 23 2008
14:41:19
Y
X
Freq_Response_1

MODE5


1
DISPLACEMENT
STEP=1
SUB =5
FREQ=268.543
DMX =19.604
ANSYS
APR 23 2008
14:42:41
Y
X
Freq_Response_1

MODE6


1
DISPLACEMENT
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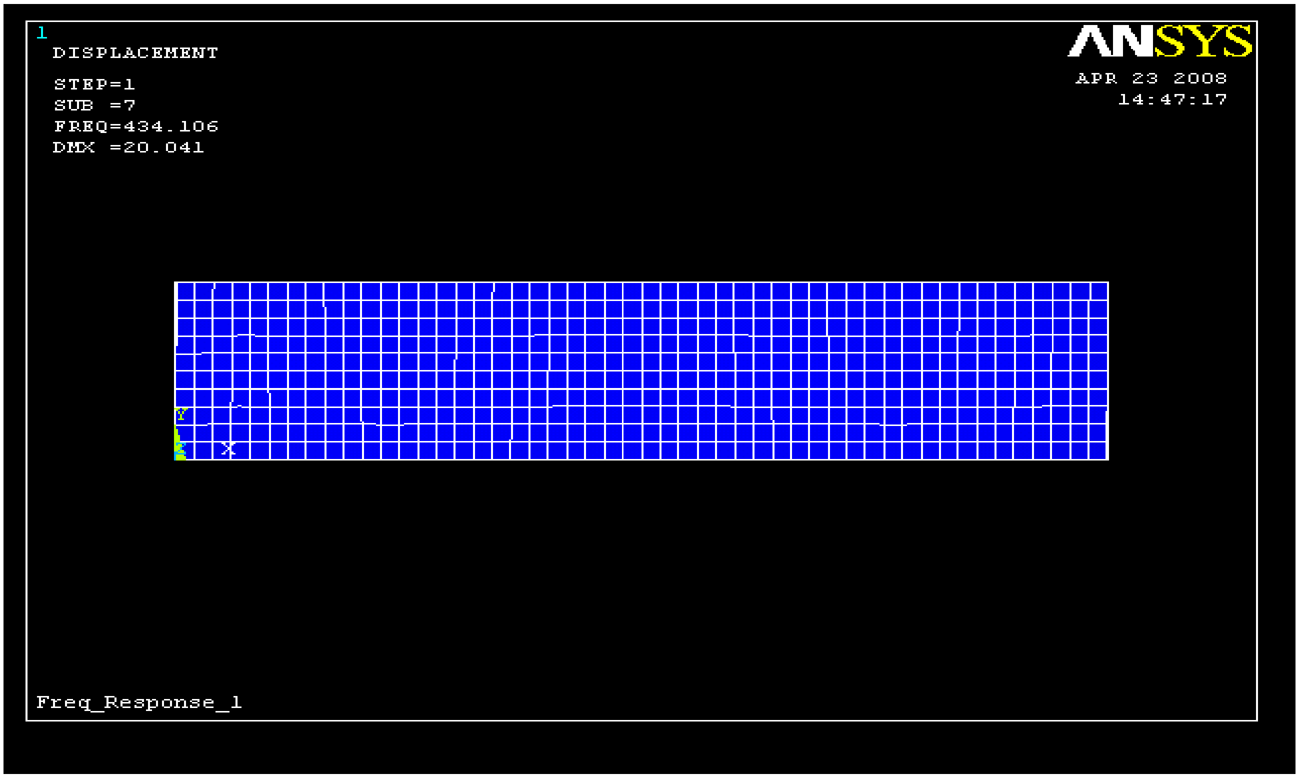
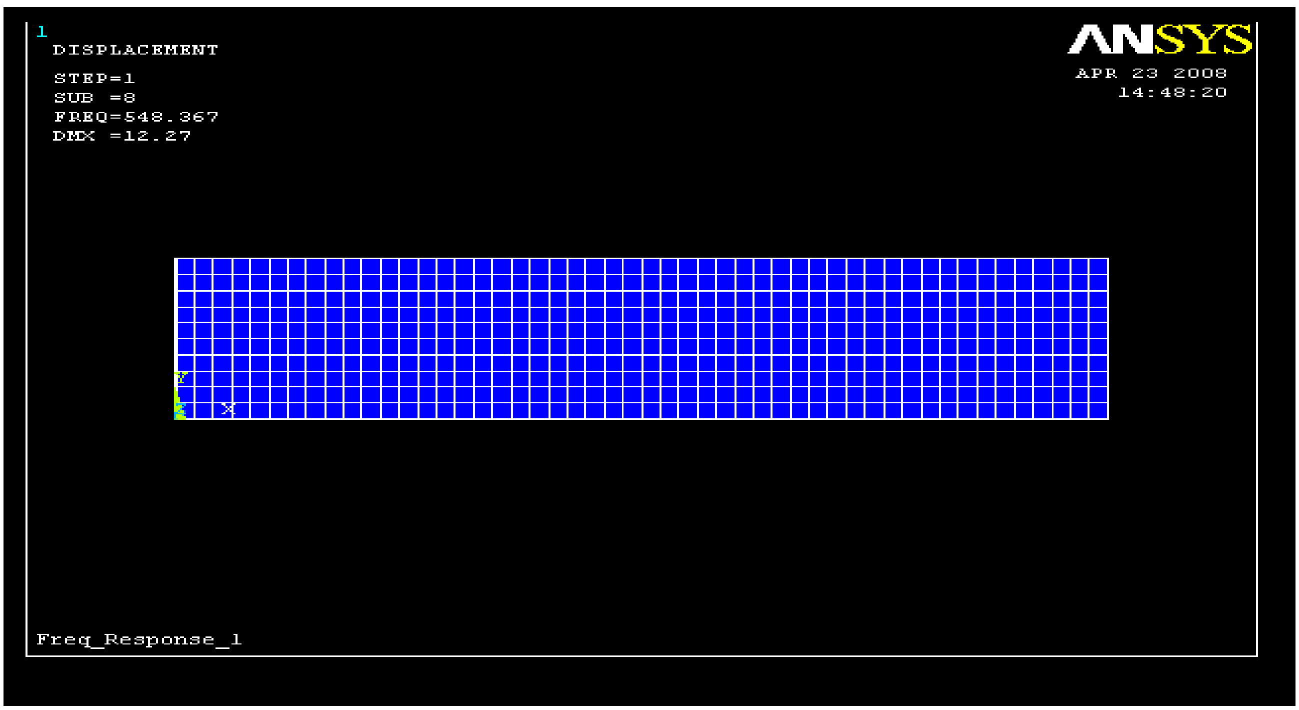
SUB =6
FREQ=330.654
DMX =11.978
ANSYS
APR 23 2008
14:45:51
Y
X
Freq_Response_1

MODE7



MODE8



6.3 Results of Fatigue Analysis

Detailed discussion and mathematical modal for prediction of fatigue behavior of composites using *S-N* curve approach. determination of residual strength at any no. of cycles (n) and any R and f. and determination of fatigue life at any R and f are been discussed in previous chapter.

6.3.1 Input Experimental S – NCurves For Different Frequency (f)

Jayantha A. Epaarachchi, Philip D. Clausen.[31]

Composite material	**Graphite/epoxy**
Laminate configuration	**$[0/90]_{8s}$ (symmetric)**
	and
	X_T = 425 Mpa

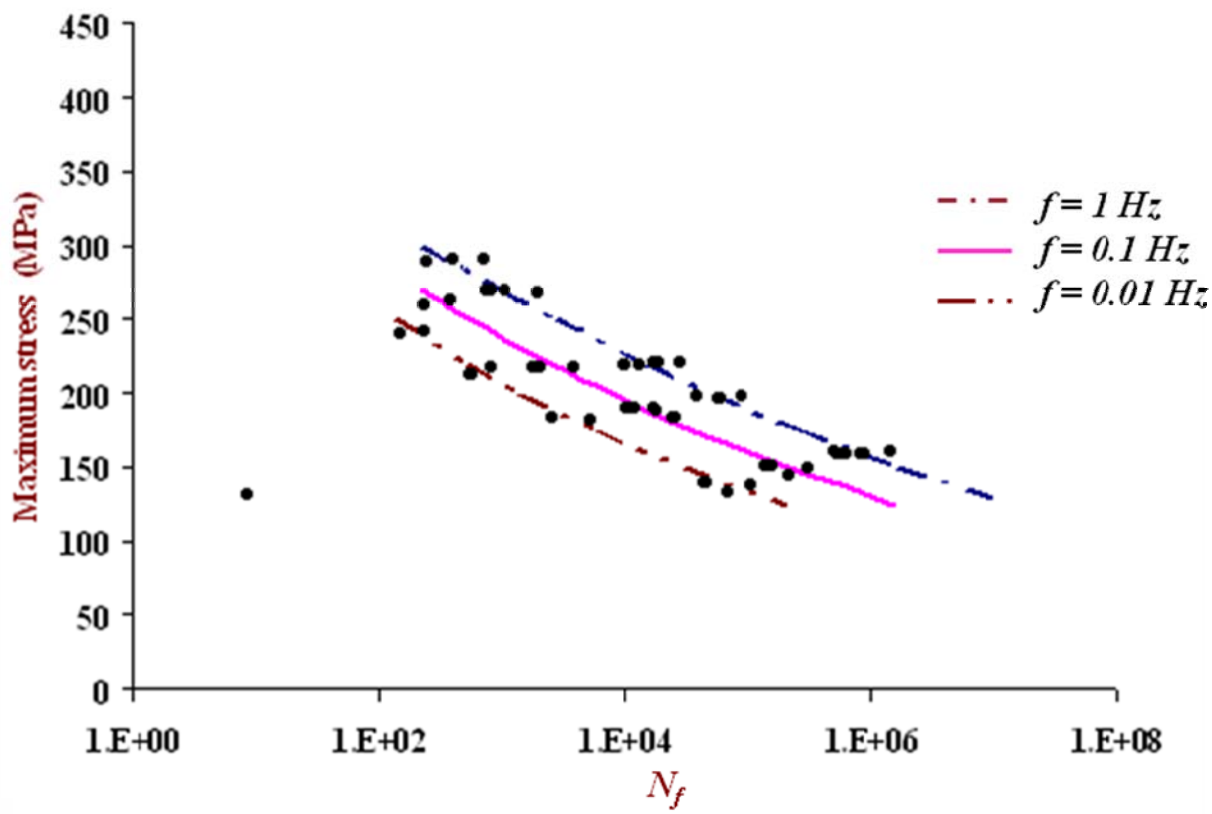


Fig. 6.1 Experimental S – N curves for different frequency (f)

Putting the values of the 3 points on the input S - N curve , σmax1 , σmax2 , σmax3 , Nf1 , Nf2 , Nf3

Data points (σ_{max} (MPa), N_f)	α	β	γ
(300, 221), (250, 2748) (200, 49391)	0.4133	0.1699	1.6471
(300,221), (270,100000) (230, 1000)	0.4215	0.1706	1.6624
(216, 10000), (253, 1000) (200, 49391)	0.4092	0.1644	1.6415

As we seen above for the same material with variation in data points α, β, γ remain same

Lets take material parameters: α =0.4133

β = 0.1699

γ = 1.6471

6.3.2 Output Maximum Stress (σ_{max}) Vs Fatigue Life (N_f)

Keeping frequency (f) constant 10Hz and Varying stress ratio (R)

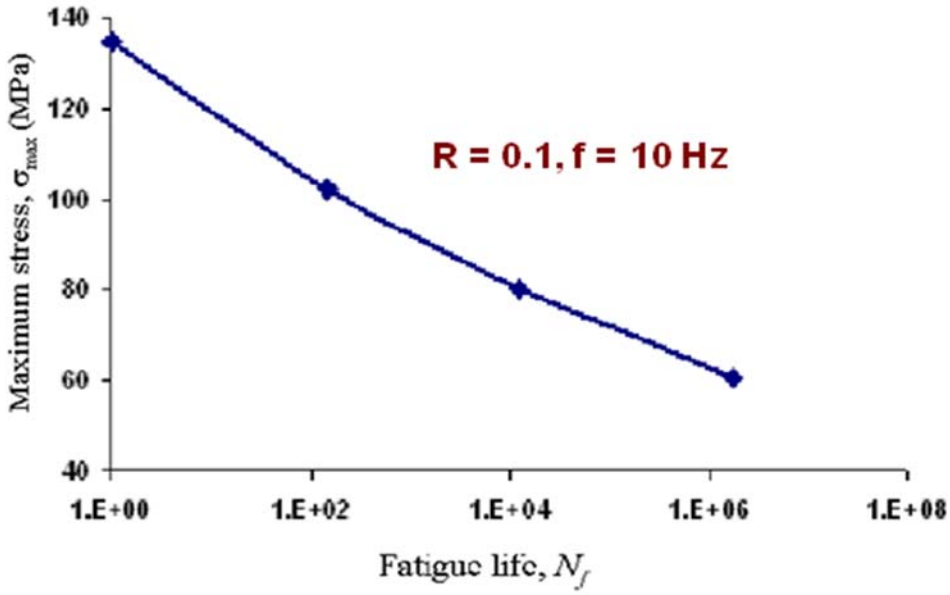


Fig. 6.2 Maximum Stress (σ_{max}) Vs Fatigue Life (N_f) For Frequency (f) 10Hz And Stress Ratio(R) 0.1

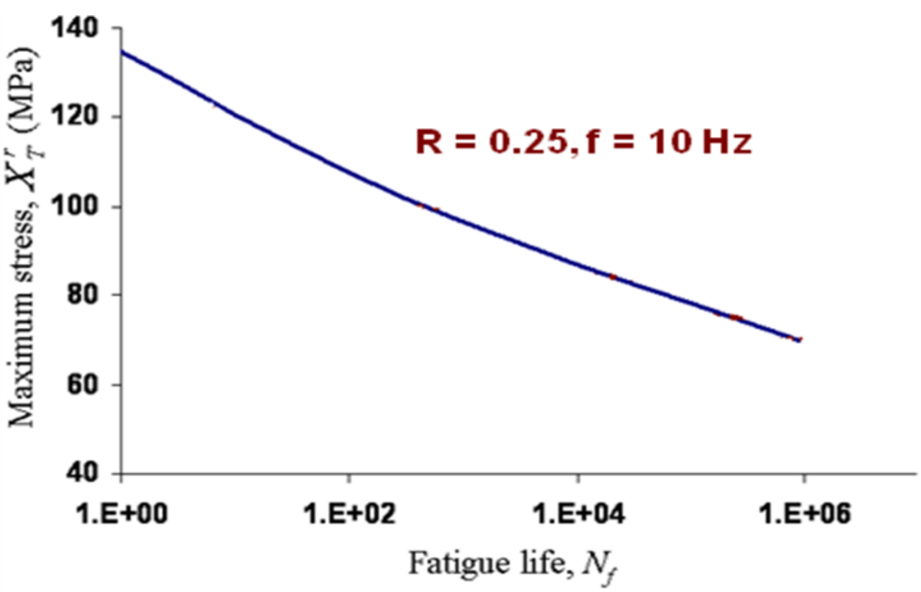


Fig. 6.3 Maximum stress (σmax) Vs Fatigue Life (Nf) for Frequency (f) 10Hz And Stress ratio (R) 0.25

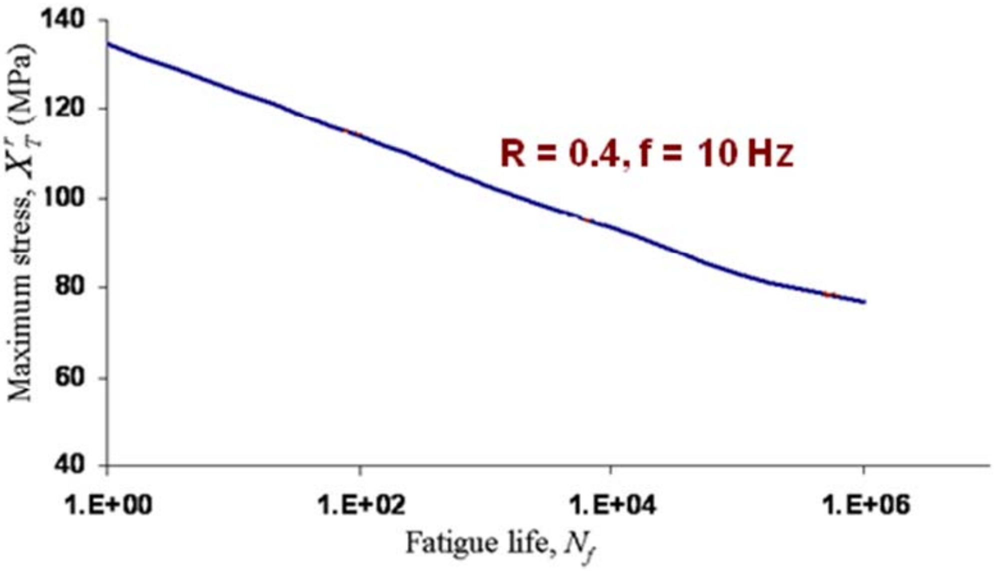


Fig. 6.4 Maximum Stress (σmax) Vs Fatigue Life (Nf) For Constant Frequency (f) 10Hz And Stress Ratio(0.4)

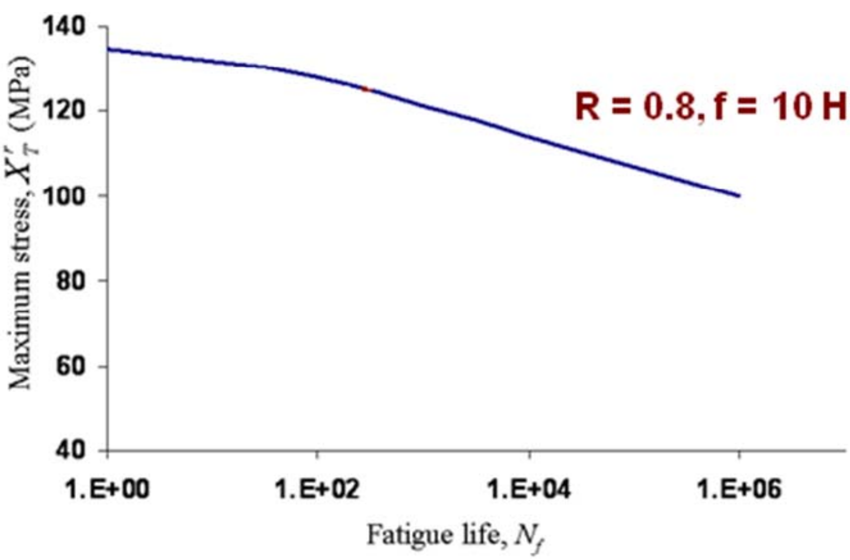


Fig. 6.5 Maximum Stress (σmax) Vs Fatigue Life (Nf) For Constant Frequency (f) 10Hz And Stress Ratio(0.8)

6.3.3 Harmonic Analysis through ANSYS 10

6.3.3(a) Input Material Properties for Composite Plates

Mohamad S. Qatu[26]

Table 6.5

Composite material	E-glass/epoxy (E/E)
Axial Young's modulus of lamina (E_1)	98 Gpa
Transverse Young's modulus of lamina (E_2)	7.9 Gpa
Axial Shear Modulus of lamina (G_{12})	5.16 Gpa
Poisson's ratio (ν)	0.28
Dimensions	a=304.8 mm, b=76.2 mm,
h/ply	0.134 mm
Density of lamina (ρ)	1520 kg/m3
Load	100 N
Frequency of loading	19 Hz

6.3.3(b) Loads Acting Along Fiber Direction

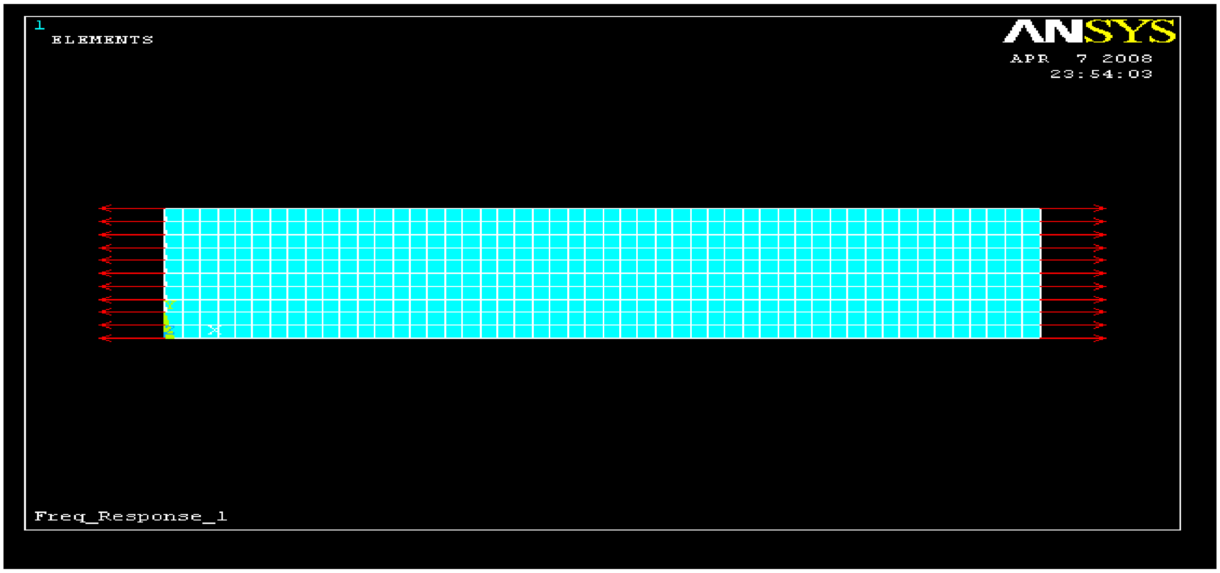


Fig.6.6 When Loads With Frequency 19Hz Along The Fiber Direction

Obtained values are

Min. stress $0.139*10^{7}$ N/m^{2}

Max. stress $0.579*10^{7}$ N/m^{2}

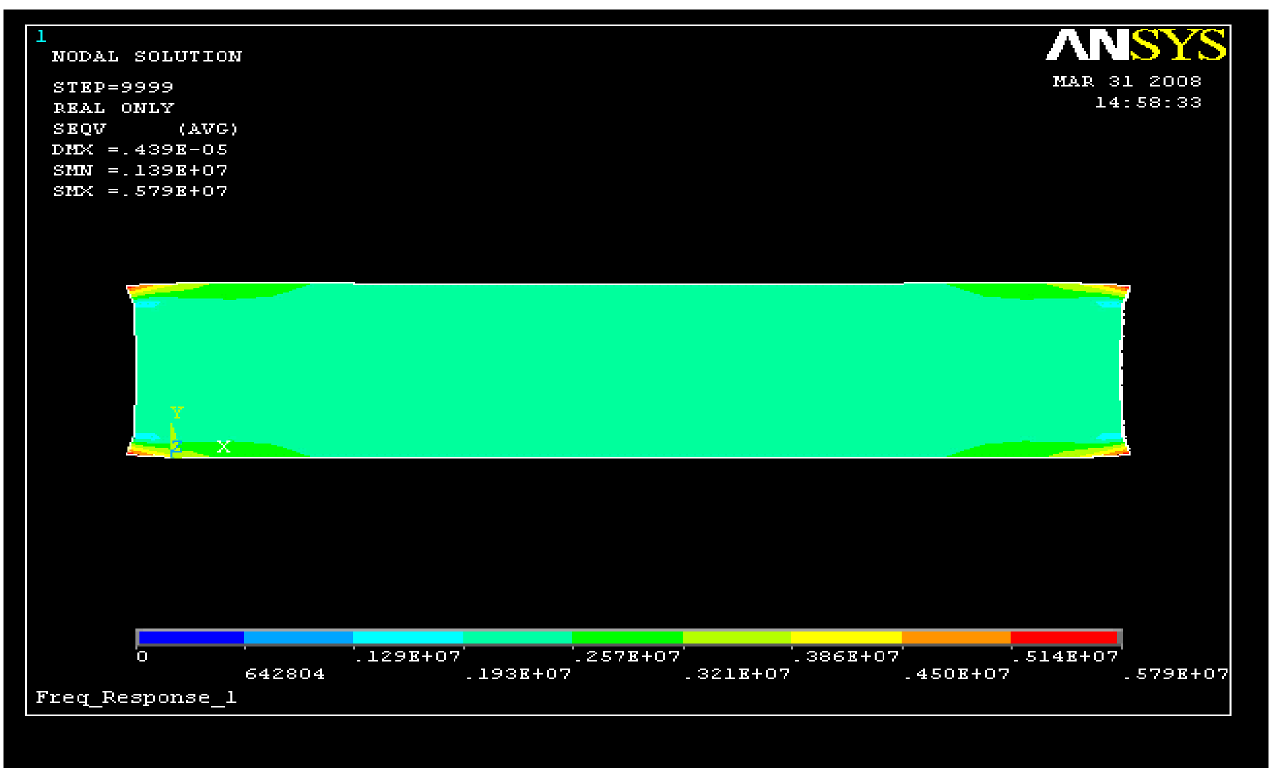


Fig.6.7 Maximum And Minimum Stress Value

Min. displacement $0.303*10^{-8}$ m

Max. displacement $0.439*10^{-5}$ m

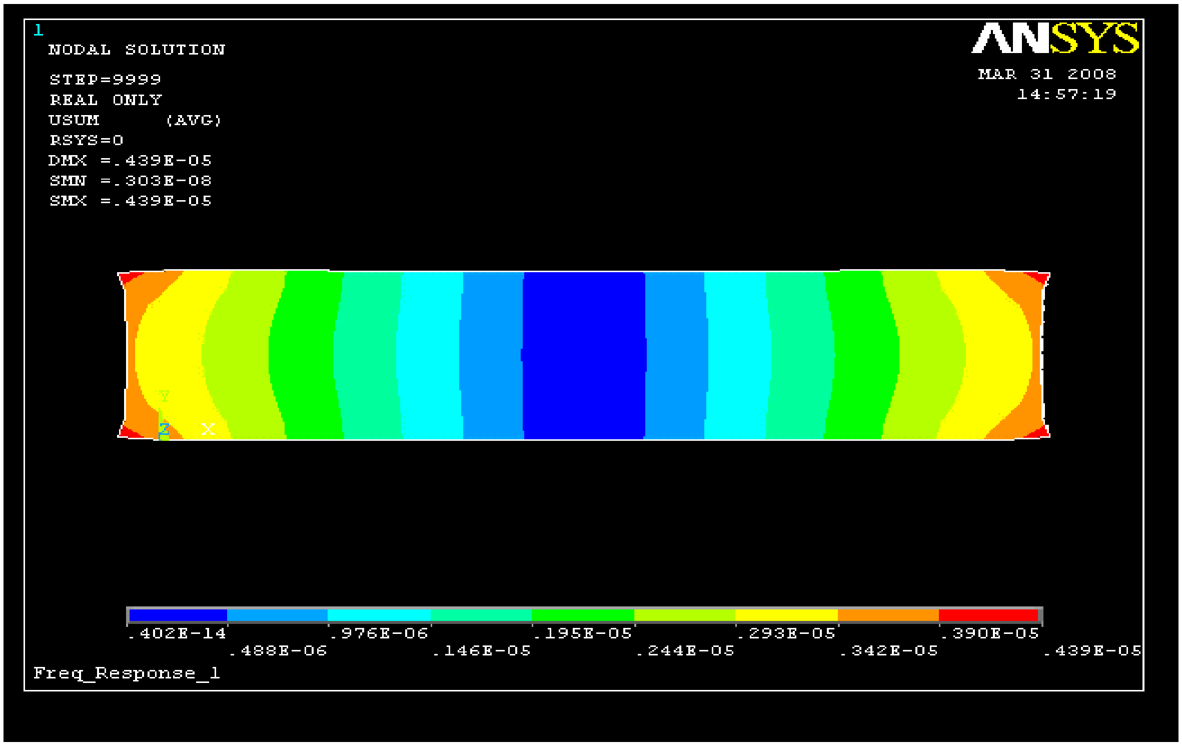
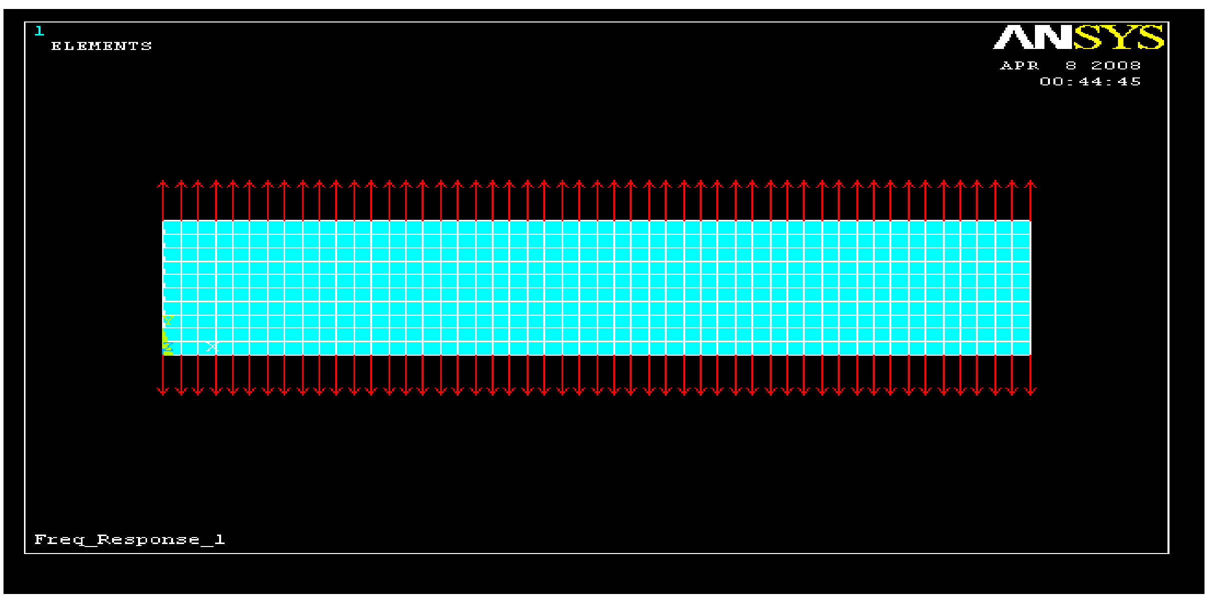


Fig.6.8 Maximum And Minimum Displacemens Value

6.3.3(c) Loads Acting Transverse To Fiber Direction



. **Fig. 6.9** When Loads With Frequency 19Hz Transverse To Fiber Direction

Obtained values are

Min. stress 77282 N/m^2 ,

Max. stress 334757 N/m^2

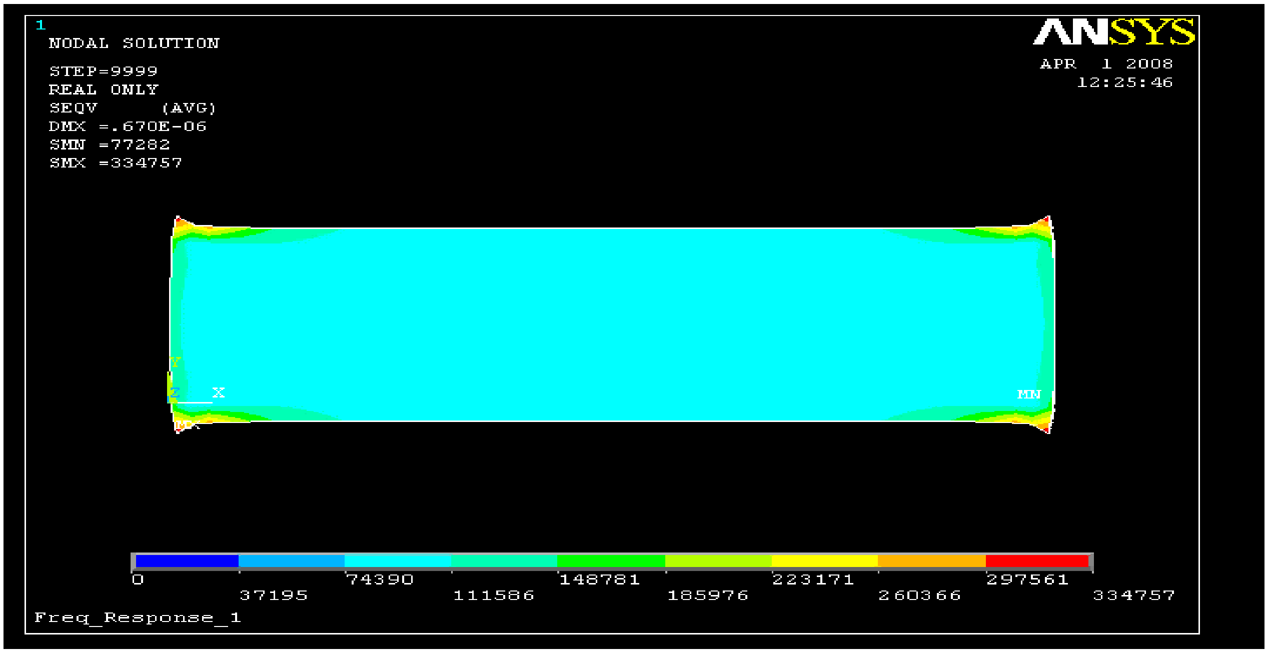


Fig.6.10 Maximum And Minimum Stress Value

Min. displacement 0.122*10^{-8} m ,

Max. displacement 0.670*10^{-6} m

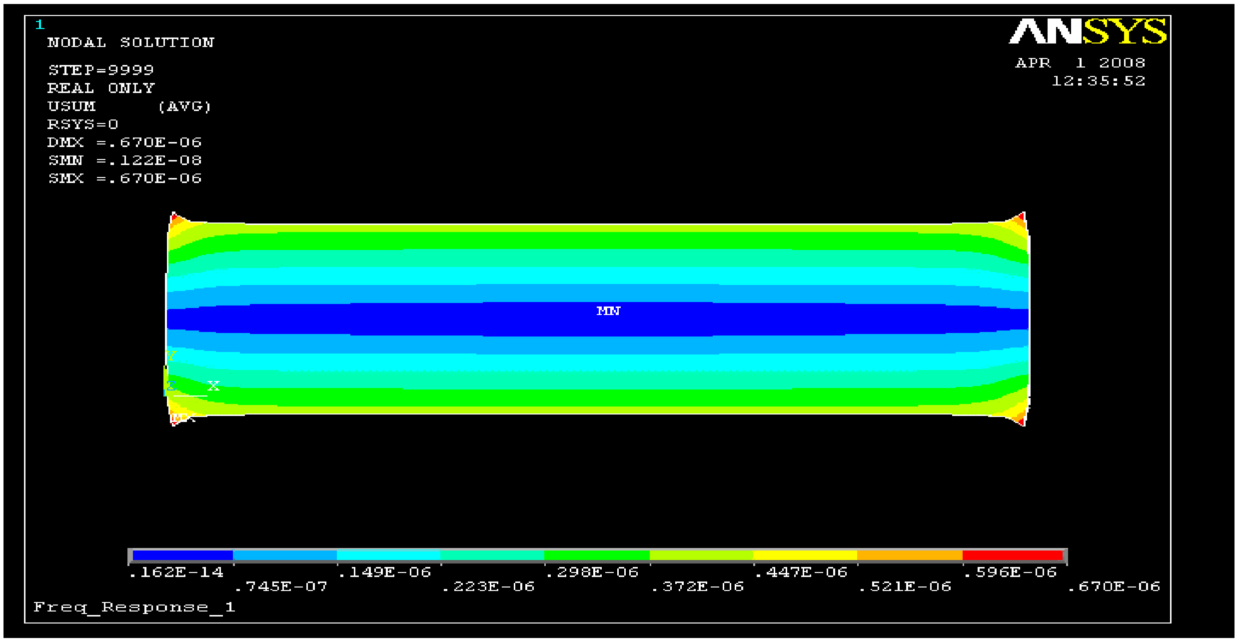


Fig.6.11 Maximum And Minimum Displacements

CHAPTER 7

CONCLUSION AND SCOPE FOR FUTURE WORK

Along with the advance of science and technology, composite materials including laminated composite plates have been widely used in various engineering field such as in aeronautic, astronautic ,auto- industries, submarine engineering ,nuclear technology ,and also for the fine construction such as circuit boards in electronic packages, thereby creating considerable interest in their analysis. In addition to their high strength / light weight, another important advantage of composite laminates is that structural properties can be tailored through changing the fiber angle and/or the number of plies .Various kinds of composite materials provides a wide range of selections for engineers. The need for more information on the behavior of laminated structural components, like plates, is clear. Rectangular plates are used in many engineering applications.

Here we are using micromechanics to predict the various properties of orthotropic/transversely isotropic lamina which includes elastic properties, thermal properties, and strength.

Many of the differential equations arising in science and engineering can be recast in the form of a matrix eigenvalue problem. Solution of this equation within the context of the Rayleigh-Ritz variation method may be viewed as one of the fundamental tasks of numerical analysis. Successive approximation approaches to the Rayleigh-Ritz problem seek to improve eigenvectors and eigenfunctions by sequentially refining a trial function.

The fatigue behavior of composite materials is conventionally characterized by S - N curve or damage mechanisms. For every new material with a new lay-up, altered constituents or different processing procedure, a whole new set fatigue life tests has to be repeated. Here an analytical method is presented that includes the effect of stress ratio and load frequency for the prediction of fatigue life and residual strength of composite structures.

Future Work

- Inverse microechanics for predicting fiber properties when orthotropic lamina properties are known.

- Fatigue analysis using S-N curve of composite structure for transverse, bending, buckling type of mechanical loading

- Determination vibration frequencies when laminated composite subjected to external forces.

- Determination of fatigue life in vibration environment itself .

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Appendix

Calculation of coefficient 'A_i'

$$A_1 = C_{22}^m\left(1+\left(h_2/h_1\right)\right) \text{ and } A_{12} = C_{22}^m\left(1+\left(h_3/h_1\right)\right)$$
$$A_2 = C_{23}^m\left(h_1/h_2\right)$$
$$A_3 = A_{11} = C_{23}^m$$
$$A_4 = C_{22}^m\left(h_1/h_2\right)+C_{22}^f \text{ and } A_9 = C_{22}^m\left(h_1/h_3\right)+C_{22}^f$$
$$A_5 = A_7 = C_{23}^f$$
$$A_8 = h_2 C_{23}^m/h_1 \text{ and } A_6 = h_3 C_{23}^m/h_1$$
$$A_{10} = h_1 C_{23}^m/h_2$$

Calculation of coefficient 'T_i'

$$D = A_1\left[A_{12}\left(A_5A_7 - A_4A_9\right)+A_6A_9A_{10}\right]+A_2\left[A_4A_8A_{12}+A_6\left(A_7A_{11}-A_8A_{10}\right)\right]$$
$$+A_3\left[A_4 A_9 A_{11}+A_5\left(A_8 A_{10}-A_7 A_{11}\right)\right]$$

$T \Rightarrow$ Constant is given as

$T_1 = \left[-\left(A_5A_8A_{12}+A_6A_9A_{11}\right)\right]/D$ $T_2 = \left[A_2A_8A_{12}+A_3A_9A_{11}-A_1A_9A_{12}\right]/D$

$T_3 = \left[A_1A_5A_{12}+A_2A_6A_{11}-A_3A_5A_{11}\right]/D$ $T_4 = \left[A_1A_6A_9+A_8\left(A_3A_5-A_2A_6\right)\right]/D$

$T_5 = \left[A_{10}A_6A_9+A_{12}\left(A_7A_5-A_4A_9\right)\right]/D$ $T_6 = \left[-\left(A_2A_7A_{12}+A_3A_9A_{10}\right)\right]/D$

$T_7 = \left[A_3A_5A_{10}+A_2\left(A_4A_{12}-A_6A_{10}\right)\right]/D$ $T_8 = \left[A_2A_6A_7+A_3\left(A_4A_9-A_5A_7\right)\right]/D$

$T_9 = \left[A_{12}A_4A_8+A_6\left(A_7A_{11}-A_8A_{10}\right)\right]/D$ $T_{10} = \left[A_1A_7A_{12}+A_3\left(A_8A_{10}-A_7A_{11}\right)\right]/D$

$T_{11} = \left[A_3A_4A_{11}+A_1\left(A_6A_{10}-A_4A_{12}\right)\right]/D$, $T_{12} = \left[-\left(A_1A_7A_6+A_3A_4A_8\right)\right]/D$

$T_{13} = \left[A_4A_9A_{11}+A_5\left(A_8A_{10}-A_7A_{11}\right)\right]/D$, $T_{14} = \left[A_1A_9A_{10}+A_2\left(A_7A_{11}-A_8A_{10}\right)\right]/D$

$$T_{15} = \left[-\left(A_1A_5A_{10} + A_2A_4A_{11}\right)\right]/D\ , \qquad T_{16} = \left[A_2A_4A_8 + A_1\left(A_5A_7 - A_4A_9\right)\right]/D$$

Calculation of shear modulus

$$G^n = \left(E^n / \left(2\times\left(1+\nu^n\right)\right)\right)$$

Calculation of stiffness matrix elements

$$k = 0.25E_l^i / \left[0.5\left(1-\nu_{lt}^i\right)\left(E_l^i/E_t^i\right) - \left(\nu_{lt}^i\right)^2\right]$$

$$C_{11}^i = E_l^i + 4k\left(\nu_{lt}^i\right)^2$$

$$C_{12}^i = 2k\nu_{lt}^i$$

$$C_{22}^i = k + \left[0.5E_t^i / \left(1+\nu_{tt}^i\right)\right]$$

$$C_{23}^i = k - \left[0.5E_t^i / \left(1+\nu_{tt}^i\right)\right]$$

$$C_{44}^i = G_{lt}^i$$

$$C_{66}^i = \left(C_{22}^i - C_{23}^i\right)/2$$

$$C_{13}^i = C_{21}^i = C_{31}^i = C_{12}^i$$

$$C_{32}^i = C_{23}^i$$

$$C_{55}^i = C_{44}^i .$$

1

Calculation of area of subcells

h_1 – height of fiber portion in cell assume $h_1 = 7\times10^{-6}m$

$$h_2 = \frac{1}{2}\frac{\left(-nV_f - V_f + \sqrt{\left(\left(nV_f\right)^2 - 2nV_f^2 + V_f^2 + 4nV_f\right)}\right)h_1}{nV_f}$$

$$h_3 = nh_2$$
$$h_4 = h_1 + h_3$$
$$h = h_1 + h_2$$

$$(a')^{11} = h_1 \times h_1$$
$$(a')^{12} = h_1 \times h_3$$
$$(a')^{21} = h_2 \times h_1$$
$$(a')^{22} = h_2 \times h_3$$
$$\Lambda' = (h_1 + h_2)(h_1 + h_3) = hh_4$$

Calcualtion of stress concentration matrix elements

This is obtained by putting $\bar{\sigma}_{11} = 1$ and other stress values zero in $\bar{\varepsilon}_{11} = S_{ij}\bar{\sigma}_{ij}$

$$\varepsilon_{11} = 1 / E_1$$
$$\varepsilon_{22} = -(\nu_{12} / E_1)$$
$$\varepsilon_{33} = -(\nu_{13} / E_1)$$

$$J_1 = hC_{22}^m \varepsilon_{22} / h_1 + hC_{23}^m \varepsilon_{33} / h_2$$
$$J_2 = (C_{12}^m - C_{12}^f)\varepsilon_{11} + hC_{22}^m \varepsilon_{22} / h_2 + hC_{23}^m \varepsilon_{33} / h_1$$
$$J_3 = (C_{12}^m - C_{12}^f)\varepsilon_{11} + h_4 C_{23}^m \varepsilon_{22} / h_1 + h_4 C_{22}^m \varepsilon_{33} / h_3$$
$$J_4 = h_4 C_{23}^m \varepsilon_{22} / h_3 + h_4 C_{22}^m \varepsilon_{33} / h_1$$
$$\psi_1 = T_1 J_1 + T_2 J_2 + T_3 J_3 + T_4 J_4$$
$$\psi_2 = T_5 J_1 + T_6 J_2 + T_7 J_3 + T_8 J_4$$
$$\psi_3 = T_{13} J_1 + T_{14} J_2 + T_{15} J_3 + T_{16} J_4$$
$$\psi_4 = T_9 J_1 + T_{10} J_2 + T_{11} J_3 + T_{12} J_4$$
$$Bs_{11}^{11} = C_{11}^f \varepsilon_{11} + C_{12}^f (\psi_1 + \psi_4)$$

Similarly other elements calculated as

$$\varepsilon_{11} = -\left(\nu_{12} / E_1\right)$$

$$\varepsilon_{22} = 1 / E_2$$

$$\varepsilon_{33} = -\left(\nu_{21} / E_2\right) \qquad \text{for } Bs_{22}^{(\beta\gamma)}$$

C programme for calculating strengths of lamina

Input- Fiber properties : E_l^f , E_t^f ,ν_{lt}^f , ν_{tt}^f , G_{lt}^f , k_l^f , k_t^f , α_l^f , X_t^f , X_c^f
Matrix properties : E^m , ν^m , α^m , k^m , Y^m , S^m
Fiber volume fraction : V_f

Output- Strength properties of lamina : X_t , Y_t , Z_t , X_c , Y_c , Z_c , S_{12} , S_{13} , S_{23}

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
int Klf,Ktf,Cf[6][6],Cm[6][6]B[6][6],E[6][6]={{0,0,0,0,0,0},
                                  {0,0,0,0,0,0},
                                  {0,0,0,0,0,0},
                                  {0,0,0,0,0,0},
                                  {0,0,0,0,0,0},
                                  {0,0,0,0,0,0}};
float
Elf,Etf,Gltf,Xtf,Xcf,Alf,Atf,Blf,Btf,Elm,Xtm,Xcm,vf,vlf,vtf,El,S
[6][6];
float Sf[6][6],Sm[6][6];
const long double h1=0.0000007;
printf("fibre->Graphite & Matrix->Polymer");
printf("input fiber properties for orthotropic lamina
calculations:");
Elf=233000;
Etf=23100;
Gltf=8960;
Xtf=2250;
Xcf=2000;
Klf=12;
Ktf=15;
Alf=-0.54;
Atf=10.10;
Elm=4620;
Xtm=60;
Xcm=200;
h2=1/2*((-n*vf)-vf+sqrt((n*vf)^2-(2*n*vf)^2+vf^2+4*n*vf));
h3=n*h2;
h4=h1+h3;
h=h1+h2;
printf("enter the values of coefficient of moisture
expansion:Blf and Btf");
scanf("%f %f",Blf,Btf);
```

```
printf("enter values of elastic stiffness matrix B");
for(i=0;i<6;i++)
{
  for(j=0;j<6;j++)
  {
    scanf("%d",b[i][j]);
  }
}
E[1][1]=B[1][1];
E[1][2]=E[1][3]=B[1][2];
E[2][2]=E[3][3]=(3/4*B[2][2])+(1/4*B[2][3])+(1/2*B[6][6]);
E[2][3]=(1/4*B[2][2])+(3/4*B[2][3])-(1/2*B[6][6]);
E[4][4]=E[5][5]=B[4][4];
E[6][6]=1/2*(E[2][2]-E[2][3]);
if(n==1)
{
for(i=0;i<5;i++)
{
  for(j=0;j<5;j++)
  {
    S[i][j]=E[j][i];
  }
}
}
else
{
for(i=0;i<5;i++)
{
  for(j=0;j<5;j++)
  {
    S[i][j]=B[j][i];
  }
}
}
e1=1/S[1][1];e2=1/S[2][2];
e3=1/S[3][3];
v12=-(S[1][2]*e1);
v13=-(S[1][3]*e1);
v23=-(S[2][3]*e2);
g12=1/S[4][4];g13=1/S[5][5];g23=1/S[6][6];
Cf[1][1]=El+4*k*(vlf^2);
Cf[1][2]=2*k*vlf
Cf[2][2]=k+(0.5*El/1+vtf);
Cf[2][3]=k-(0.5*El/1+vtf);
Cf[4][4]=Glt;
Cf[6][6]=(Cf[2][2]-Cf[2][3])/2;
Cf[1][3]=Cf[2][1]=Cf[3][1]=Cf[1][2];
```

```
Cf[3][2]=Cf[2][3];
Cf[5][5]=Cf[4][4];
for(i=0;i<5;i++)
{
for(j=0;j<5;j++)
{
Sf[i][j]=Cf[j][i];
Sm[i][j]=Cm[j][i];
Bs[i][j]=(1/v[i][j])*(Sm[i][j]-Sf[i][j]);
}
}
Bsf=(1/vf)*(Sf-Sm);
printf("for fiber Bsf=",Bsf);
Xt=Xtf/Bs[1][1];
Xc=Xcf/Bs[1][1];
printf("ultimate longitudinal strength of lamina: Xt=%f
Xc=%f",Xt,Xc);
Yt=Xt/max(Bs[1][2],Bs[2][1],Bs[2][2]);
Yc=Xc/max(Bs[1][2],Bs[2][1],Bs[2][2]);
printf("ultimate transverse stress of lamina: Yt=%f
Yc=%f",Yt,Yc);
printf("ultimate shear stress of lamina: S12=%f
S13=%f",S[1][2],S[1][3]);
S=Sm/max(Bs[1][2],Bs[2][1],Bs[2][2]);
printf("shear strength :%f",S);
getch();
}
```

MATLAB Programme to generate Residual strength Vs Number of cycles plot

---

Input - σ_{max1}, σ_{max2}, σ_{max3}, N_{f1}, N_{f2}, N_{f3}

Output - values of α β γ, Residual strength Vs Number of cycles plot for ant value of R and f

---

```
close all;
clear all;
clc;
X_T=6740;
Sigma_Max1=350;
Sigma_Max2=250;
Sigma_Max3=200;
N_F1=221;
N_F2=2748;
N_F3=49391;
R=-1;
F_B=10;

B=input('Beta Value');

Gamma1=(log10(X_T-Sigma_Max1)-log10(X_T-Sigma_Max2)-
log10((N_F1.^B)-1)+log10((N_F2.^B)-1))/(log10(Sigma_Max1)-
log10(Sigma_Max2));
Gamma2=(log10(X_T-Sigma_Max1)-log10(X_T-Sigma_Max3)-
log10((N_F1.^B)-1)+log10((N_F3.^B)-1))/(log10(Sigma_Max1)-
log10(Sigma_Max3));
B
Gamma1
Gamma2
Gamma=1.6471;
B=0.1633;
Sigma_Max=350;
n=15;
Alfa=(X_T-Sigma_Max1)*(F_B.^B)/((X_T.^(1-
Gamma1))*(Sigma_Max1.^Gamma)*((1-R).^Gamma)*((N_F1.^B)-1));
Alfa
%Alfa=0.4233;
R=-10:1;
F=10:100;
Nf=((((X_T-Sigma_Max).*(F_B.^B))./(Alfa.*(X_T.^(1-
Gamma)).*(Sigma_Max.^Gamma).*((1-R).^Gamma)))+1).^(1/B);
X_T_R=(X_T)-Alfa.*(X_T.^(1-Gamma1)).*(Sigma_Max1.^Gamma).*((1-
R).^Gamma).*(1./F_B).*(n.^B-1);
Nf
X_T_R
plot(Nf,X_T_R)
```

C programme for typical successive approximations approach to the eigenvalue Problem

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
#include<string.h>
 #Include<stdlib.h>
void main()
{

/*
   LINEAR RAYLEIGH-RITZ ALGORITHM

   To approximate the solution of the boundary-value problem

                D(P(X)Y")/DX + Q(X)Y = F(X), 0 <= X <= 1,
                Y(0) = Y(1) = 0,

                with a piecewise linear function:

    INPUT:    integer N; mesh points X(0) = 0 < X(1)
                                      < X(N) < X(N+1) = 1

    OUTPUT:   coefficients C(1),...,C(N) of the basis functions
*/
     Float X[26],H[25],A,B,ALPHA,BETA,ZETA,Z,C[25],Q[6][26];
     Float HQ,HC;
       int   N1,J1,FN,N,J,FLAG;
       char AA ;
       char NAME[30];
  //    INP,OUP : text;
  ///*  Change functions P, QQ and F for a new problem
                                                        */
             float P(float X )
            {
             P = 1.0
                             };
             float  QQ( float X )
           {
             QQ = PI * PI
                              };
             float F ( float X )
           {
             F = 2.0 * PI * PI * sin( PI * X )
                                            };
```

```
void  INPUT();
{
  printf("This is the Piecewise Linear Rayleigh-Ritz
  Method.");
  printf ("This program requires functions P, QQ, F to be
  created and ");
  printf ("X(0), ..., X(N+1) to be supplied. ");
  printf ("Are the preparations complete? Answer Y or N. ");
  scanf( "%f",&AA );
  if ( ( AA == "Y" ) or ( AA == "y" ) ) then
     {
        OK = false;
        while ( !OK )
           {
           printf("Input integer N where X(0) = 0, X(N+1)
                                                   = 1.");
           scanf( N );
           if ( N < 2 ) then printf ("N must be greater
                                                  than 1. ")
              else OK = true;
           };
        X[0] = 0.0;
        X[N+1] = 1.0;
        printf ("Choice of method to input X(1), ..., X(N):
                ");
        printf ("1.  Input from keyboard at the prompt ");
        printf ("2.  Equally spaced nodes to be calculated
                ");
        printf ("3.  Input from text file ");
        printf ("Please enter 1, 2, or 3. ");
       scanf( "%d",&FLAG );

     if ( FLAG == 2 ) then
          {
             HC = 1.0 / ( N + 1.0 );
             for (J = 1;J<= N1;J++)

          {
                 X[J] = J * HC;
                 H[J-1] = HC ;
                                         }
                 H[N] = HC
                                         }
                 else
          {
                 if ( FLAG == 3 ) then
```

```
   {
    printf ("Has the input file been created?");
    printf (" - enter Y or N. ");
    scanf( "%f",&AA );
  if ( AA = "Y" ) or ( AA = "y" ) then
     {
   printf("Input the file name in the form -");
   printf(" drive:name.ext, ");
   printf ("for example: A:DATA.DTA ");
   scanf( "%s",NAME );

   for (J = 1;J<= N1;J++) read ( INP, X[J] );
   for (J = 0;J<= N1;J++) H[J] = X[J+1] - X[J];
   close ( INP )
                              }
  else
  {
   printf ("The program will } so");
   printf(" the input file can be created. ");
   OK = false;
               }
               }
  else
  {
  for (J = 1;J<= N1;J++){
   printf ("Input X(",J,"). ");
   scanf( X[J] );
   H[J-1] = X[J] - X[J-1]
                              };
   H[N] = X[N+1] - X[N]
                             }
                             }
                             }
else
{
   printf ("The program will } so that the functions ");
   printf ("can be created. ");

           OK = false
                          }
                          };

   float SIMPSON( int FN , float  A, float B )
                                             /*
       {
      float Z[4],Y,H ;
      int I;
```

```
            H = ( B - A ) / 4.0;
            for (I = 0:I<=4;I++)
         {
            Y = A + I * H;
            switch(FN)
         {
       case 1: Z[I] = ( 4.0 - I ) * I * sqrt( H ) * QQ(Y);
       case 2: Z[I] = sqrt( I * H ) * QQ( Y );
       case 3: Z[I] = sqrt( H * ( 4.0 - I ) ) * QQ( Y );
       case 4: Z[I] = P( Y );
       case 5: Z[I] = I * H * F( Y );
       case 6: Z[I] = ( 4.0 - I ) * H * F( Y )
                                            }
                                            };
    SIMPSON = ( Z[0] + Z[4] + 2.0 * Z[2] + 4.0 * ( Z[1] + Z[3]
              ) ) *  H / 3.0;
                                };
void main()
 {
    printf ("Choice of output method: ");
    printf ("1. Output to screen ");
    printf ("2. Output to text file ");
    printf ("Please enter 1 or 2.");
    scanf("%d",&FLAG );
    if ( FLAG = 2 ) then
       {
          printf ("Input the file name in the form -
          drive:name.ext, ");
          printf("for example:   A:OUTPUT.DTA");
          scanf( "%s",NAME );
          strcpy ( OUP, NAME )
                               }
    else strcpy ( OUP, "CON" );
    rewrite ( OUP );
    printf(OUP,"PIECEWISE LINEAR RAYLEIGH-RITZ METHOD");

 printf(OUP);
 printf (
 OUP,"I":3,"X(I1)":12,"X(I)":12,"X(I+1)":12,"C(I)":14);
 printf ( OUP );
        for J = 1 to N do
 printf(OUP,J:3,X[J- 1]:12:8,X[J]:12:8,X[J+1]:12:8,C[J]:14:8);
    close ( OUP )
              };
```

```
{
      INPUT;
   /*     Step 1 is done within the input procedure    */

          if ( OK ) then
        {
          N1 = N - 1;
               /*       STEP 3  */

 for (J = 1;J<= N1;J++)                  {
         Q[1,J] = SIMPSON( 1, X[J], X[J+1] ) / sqrt( H[J] );
         Q[2,J] = SIMPSON( 2, X[J-1], X[J] ) / sqrt( H[J-1] );
         Q[3,J] = SIMPSON( 3, X[J], X[J+1] ) / sqrt( H[J] );
         Q[4,J] = SIMPSON( 4, X[J-1], X[J] ) / sqrt( H[J-1] );
         Q[5,J] = SIMPSON( 5, X[J-1], X[J] ) / H[J-1] ;
         Q[6,J] = SIMPSON( 6, X[J], X[J+1] ) / H[J]
                                                          };
         Q[2,N] = SIMPSON( 2, X[N-1], X[N] ) / sqrt( H[N-1] );
         Q[3,N] = SIMPSON( 3, X[N], X[N+1] ) / sqrt( H[N] );
         Q[4,N] = SIMPSON( 4, X[N-1], X[N] ) / sqrt( H[N-1]
                                                             );
          Q[4,N+1] = SIMPSON( 4, X[N], X[N+1] ) / sqrt( H[N]
                                                             );
          Q[5,N] = SIMPSON( 5, X[N-1], X[N] ) / H[N-1] ;
          Q[6,N] = SIMPSON( 6, X[N], X[N+1] ) / H[N] ;
          /*           STEP 4
          */
          for (J = 1;J<= N1;J++)
             {
               ALPHA[J] = Q[4,J]+Q[4,J+1]+Q[2,J]+Q[3,J];
               BETA[J] = Q[1,J]-Q[4,J+1];
               B[J] = Q[5,J]+Q[6,J]
                                  };


             /*       STEP 5     */

          ALPHA[N] = Q[4,N]+Q[4,N+1]+Q[2,N]+Q[3,N];
          B[N] = Q[5,N]+Q[6,N];
         /*       STEP 6   */

/* STEPS 6-10 solve a symmetric tridiagonal linear system
 using Algorithm */

         A[1] = ALPHA[1];
         ZETA[1] = BETA[1] / ALPHA[1];
         Z[1] = B[1] / A[1];
```

```
          /*          STEP 7        */
for (J = 2;J<= N1;J++)
{
     A[J] = ALPHA[J] - BETA[J-1] * ZETA[J-1];
     ZETA[J] = BETA[J] / A[J];
     Z[J] = ( B[J] - BETA[J-1] * Z[J-1] ) / A[J];
                                                   };
          /*         STEP 8         */
     A[N] = ALPHA[N] - BETA[N-1] * ZETA[N-1];
     Z[N] = ( B[N] - BETA[N-1] * Z[N-1] ) / A[N];

          /*         STEP 9         */

     C[N] = Z[N];
           /*      STEP 10          */

for (J = 1;J<= N1;J++)
{
      J1 = N - J;
      C[J1] = Z[J1] - ZETA[J1] * C[J1+1]
                                                 };
      OUTPUT
             }
```