Mathematical Modeling and Analysis of Fingero-Imbibition phenomenon in vertical downward cylindrical homogeneous porous matrix

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***Abstract*-- This paper focuses on the compound phenomenon of Instability and Imbibition called Fingero-Imbibition phenomenon arising in two immiscible phase (oil and water) flow through homogeneous porous media in vertical downward direction. Mathematical formulation leads to non linear partial differential equation. The analytical solution is obtained by using generalised separable method in terms of quadratic polynomial by using appropriate boundary conditions and its physical interpretation is given with numerical tabulated values and graphical presentation.**

Key words: *Instability, Imbibition, Fingero-Imbibition, Immiscible, Homogeneous porous medium, Generalised separable method*

1. INTRODUCTION

This paper discusses the compound phenomenon of Instability and Imbibition, (Scheidegger A. E. [1]) called fingero-imbibition phenomenon arising in two phase (oil and water) flow through homogeneous porous media in vertical downward direction under capillary pressure.

Fingero-imbibition phenomenon occurs during secondary recovery process when water is injected in oil formatted region to push oil towards oil reservoir. First, imbibition phenomenon occurs near the common interface due to the contact of the two immiscible fluids. Then due to external injecting force, water shoots through the oil formation and gives rise to protuberance (fingers). Thus two phenomenon Fingering and Imbibition occur simultaneously and it describes Fingero-Imbibition phenomenon.

In real, field area is too large for practical as well as experimental study. Therefore it is necessary to develop mathematical model by selecting its small part as cylindrical porous matrix. For study of one dimensional case, we take vertical cross sectional area which is rectangular shape and instead of real fingers occurs in irregular shape, it has been considered as rectangular fingers. Its average value of cross sectional area is considered for the study of saturation of injected fluid.

As shown in figure (1), let the medium be completely saturated with a non-wetting fluid (oil) and some wetting fluid (Water) is injected on top of its surface in vertical downward direction. In this situation, there will be spontaneous flow of wetting fluid into the medium, which will displaced the non-wetting fluid (oil) in downward direction under effect of gravitational force and capillary pressure.

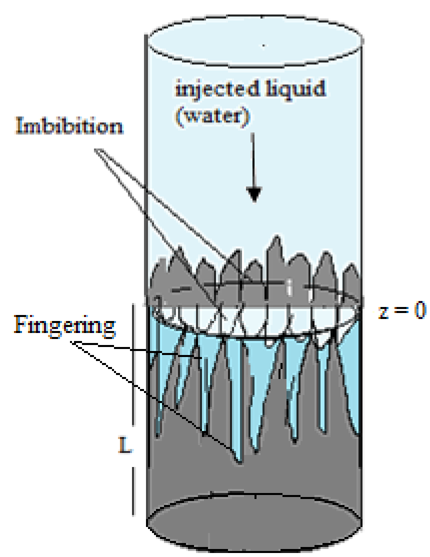

Figure: 1

Many authors have discussed this phenomenon from different point of view. Verma[2] has discussed a problem of the simultaneous occurrence of fingering and imbibition in artificial replenishment of groundwater through cracked porous medium by considering the effect of capillary pressure and variation in phase density and he was the first researcher who has designated this compound phenomenon as fingero-imbibition. Other authors such as Scheidegger [3], Graham and Richardson [4], Mehta [5] have contributed in the study of this phenomenon. Recently, Patel and Mehta [6] have derived the solution of partial differential equation representing the fingero-imbibition phenomenon by converting it into Burger's equation. Yadav and Mehta [7] discussed similarity solution of fingero-imbibition

978-1-4673-1719-1/12/$31.00©2013IEEE

phenomenon in banded porous matrix. Meher and Mehta [8] have applied Adomian decomposition method to find the solution of partial differential equation representing the fingero-imbibition phenomenon in double phase flow through porous media. Kinjal and Mehta [9] have given power series solution for this phenomenon in heterogeneous porous media. In fact, most of researchers have worked on fingero-imbibition phenomenon in horizontal direction, but very few efforts have been made for this phenomena occurring in vertical downward direction. Parikh et.al [10] have discussed counter current imbibition phenomenon in vertical downward direction and obtained analytical solution using generalized separable method. In continuous efforts, the present paper analytically discusses a fingero-imbibition phenomenon in vertical downward direction in homogeneous porous media.

2. STATEMENT OF THE PROBLEM

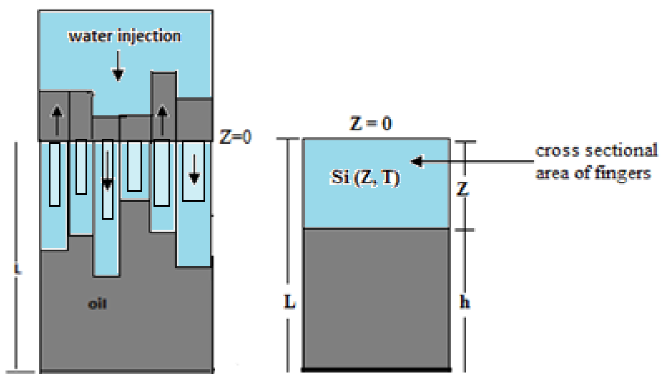

Figure: 2

To understand this phenomenon with mathematical point of view, here it is considered that a finite cylindrical piece of homogeneous porous matrix of length L is fully saturated with native fluid (oil). It is completely surrounded by an impermeable surface except top of the cylinder, which is labelled as imbibition surface (z=0) & this end is exposed to an adjacent formation of the injected fluid (water), Figure (1). Since water preferentially wets the medium it gives rise to counter-current imbibitions first that is spontaneous linear flow of wetting fluid (water) into the medium and a counter flow of the resident fluid (oil) from the medium. Hence fingers are generated of very small size up to distance x = *l* which is very near to common interface. Then due to external injecting force together with gravitational effect, the size of fingers can be extend up to the end of porous matrix at x=L. Thus two phenomenons Fingering and Imbibition occur simultaneously and it describes Fingero-Imbibition phenomenon. This phenomenon is shown in the figures (1) and (2). For study of one dimensional case we take vertical cross sectional area which is rectangular shape, Figure (2). Actually the fingers are generated in irregular shape so instead of real fingers occurs in irregular shape, it has been considered as rectangular fingers as shown in schematic figure (2) and its average length is considered for the study of saturation of injected fluid. During secondary oil recovery process when fingero-imbibition phenomenon occurs then our purpose is to measure the saturation of injected fluid (water) occupied by cross sectional area of schematic fingers of average length at any depth z for any time t > 0.

3. MATHEMATICAL MODEL

3.1 Dependent and Independent variables

In this mathematical model, the saturation $S_i(z,t)$ of injected fluid depends on depth z and time t, therefore saturation S_i of injected fluid is considered as dependent variable, depth z and time t are independent variables.

3.2 Basic assumptions

For mathematical model, the following assumptions are made:

Medium is Homogeneous. Two fluids are immiscible and injected fluid is less viscous as well as preferentially wets the medium. Flow is considered in one dimension only. In real, oil formatted region is too large. Therefore to develop mathematical model we have selected its small part as cylindrical porous matrix surrounded by impermeable surfaces except one end which is designated as common interface (Z=0) and this end is exposed to an adjacent formulation of injected fluid. For study of one dimensional case we take vertical cross sectional area which is rectangular shape and instead of real fingers occurs in irregular shape, it has been considered as regular small rectangular fingers. Its average length is considered for the saturation of injected fluid. Darcy's law is valid for oil-water flow for law Reynolds number. Macroscopic behaviour of the fingers is governed by statistical treatment. The average cross-sectional area occupied by the fingers considered by ignoring the shape and size of the fingers.

3.3 Governing laws and some standard relations

For mathematical formulations of the model some useful governing laws and standard relations are mentioned below:

(i) Since water and oil are flowing through porous medium for small Reynolds number than Darcy's law is valid in such case.

According to Bear J. and A.H.D.Cheng[11], Muskat [12], Scheidegger[1] and using Darcy's law, when water is injected in downward direction then the velocity of injected water (V_i) and velocity of oil (V_n) upward above common interface under gravitational effect will be

$$V_i = -\left(\frac{K_i}{\delta_i}\right) K \left(\frac{\partial P_i}{\partial z} + \rho_i g\right) \quad (1)$$

$$V_n = -\left(\frac{K_n}{\delta_n}\right) K \left(\frac{\partial P_n}{\partial z} + \rho_n g\right) \quad (2)$$

Where K is the permeability of the homogeneous medium, K_i and K_n are relative permeabilities of water and oil which are functions of saturations S_i and S_n, P_i and P_n are pressure, ρ_i and ρ_n are the density, δ_i and δ_n are constant kinematic viscosity of water and oil respectively, g is the acceleration due to gravity. The coordinate z is measured along the vertical axis of cylindrical medium, the origin being at the imbibition surface.

(ii) The law of Darcy is not sufficient to determine a given flow problem, In addition, there is a continuity equation of the wetting phase

$$P\left(\frac{\partial S_i}{\partial t}\right)+\frac{\partial V_i}{\partial z}=0 \tag{3}$$

where P is the porosity of the medium which is constant for homogeneous porous medium.

(iii) The capillary pressure (P_c) is defined as the pressure difference of the flowing phase across their common interface is a function of phase saturation. It may be written as

$$P_c = P_n - P_i = f(S_i) \text{ 'say'} \tag{4}$$

(iv) For definiteness, we may consider the following relationships for relative permeability-phase saturation and capillary pressure-phase saturation due to Schiedegger and Johnson [3].

$$k_i = S_i \quad , \quad k_n = 1-\alpha S_i \quad (\alpha = 1.11) \tag{5}$$

(v) As Mehta [13] suggested that capillary pressure is proportional to saturation of injected fluid in opposite direction,

$P_c = -\beta S_i$,where β is constant of proportionality (6)

3.4 Mathematical Formulations

For counter-current imbibition phenomenon, the sum of the velocities of injected fluid (water) and native fluid (oil) is zero, therefore the following imbibition condition holds true:

$V_i = -V_n$, Scheidegger [1] (7)

From (1) and (2),

$$\left(\frac{K_i}{\delta_i}\right)K\left(\frac{\partial P_i}{\partial z}+\rho_i g\right)+\left(\frac{K_n}{\delta_n}\right)K\left(\frac{\partial P_n}{\partial z}+\rho_n g\right)=0 \tag{8}$$

From equations (8) and (4)

$$\left(\frac{K_i}{\delta_i}+\frac{K_n}{\delta_n}\right)\frac{\partial P_i}{\partial z}+\left(\frac{K_n}{\delta_n}\right)\frac{\partial P_c}{\partial z}=-\left(\frac{K_i}{\delta_i}\rho_i+\frac{K_n}{\delta_n}\rho_n\right)g \tag{9}$$

$$\frac{\partial P_i}{\partial z}=-\left[\frac{\left(\frac{K_i}{\delta_i}\rho_i+\frac{K_n}{\delta_n}\rho_n\right)g+\left(\frac{K_n}{\delta_n}\right)\frac{\partial P_c}{\partial z}}{\frac{K_i}{\delta_i}+\frac{K_n}{\delta_n}}\right] \tag{10}$$

On substituting the value of $\frac{\partial P_i}{\partial z}$ from equation (10) into (1), we get

$$V_i=-\left(\frac{K_i}{\delta_i}\right)K\left[\frac{\left(\frac{K_n}{\delta_n}\right)(\rho_i-\rho_n)g-\left(\frac{K_n}{\delta_n}\right)\frac{\partial P_c}{\partial z}}{\left(\frac{K_i}{\delta_i}+\frac{K_n}{\delta_n}\right)}\right] \tag{11}$$

On substituting the value of V_i from equation (11) to (3),

$$P\left(\frac{\partial S_i}{\partial t}\right)+\frac{\partial}{\partial z}\left[K\left(\frac{\frac{K_i}{\delta_i}\frac{K_n}{\delta_n}}{\frac{K_i}{\delta_i}+\frac{K_n}{\delta_n}}\right)\frac{dP_c}{dS_i}\frac{\partial S_i}{\partial z}\right]-\frac{\partial}{\partial z}\left[K(\rho_i-\rho_n)g\left(\frac{\frac{K_i}{\delta_i}\frac{K_n}{\delta_n}}{\frac{K_i}{\delta_i}+\frac{K_n}{\delta_n}}\right)\right]=0 \tag{12}$$

This equation (12) is a non linear partial differential equation which describes the fingero-imbibition phenomenon of two immiscible fluids flow through cylindrical homogeneous porous medium with impervious bounding surface on three sides.

Since the present investigation involves water and viscous oil, therefore according to Schiedegger [1], we have

$$\left[\frac{\frac{K_i}{\delta_i}\frac{K_n}{\delta_n}}{\frac{K_i}{\delta_i}+\frac{K_n}{\delta_n}}\right]\approx\frac{K_n}{\delta_n}=\frac{1-\propto S_i}{\delta_n} \tag{13}$$

On substituting values from (12) and (6) into equation (12), we get

$$P\left(\frac{\partial S_i}{\partial t}\right)=\frac{K\beta}{\delta_n}\frac{\partial}{\partial z}\left[(1-\propto S_i)\frac{\partial S_i}{\partial z}\right]+\frac{Kg(\rho_i-\rho_n)}{\delta_n}\frac{\partial}{\partial z}(1-\propto S_i) \tag{14}$$

This is the desired non linear partial differential equation describing Fingero-Imbibition phenomenon.

To solve this equation, we choose the following appropriate initial and boundary conditions as follows:

Let the Initial Saturation of injected fluid is

$$S_i(z,0)=S_0(z) \tag{15}$$

Let saturation of injected fluid at common interface (z=0) is

$$S_i(0,t)=S_{i0} \ , \qquad t>0 \tag{16}$$

Saturation of injected fluid at the bottom of cylindrical porous matrix (z=L) is

$$S_i(L,t)=S_{i1} \ , \qquad t>0 \tag{17}$$

where L is the total length of cylindrical porous matrix, S_{i0} *and* S_{i1} are the saturations at z = 0 and $z = L$ respectively.

For more simplification, putting $S=1-\propto S_i$ in equation (14), we get

$$P\left(\frac{\partial S}{\partial t}\right)=\frac{K\beta}{\delta_n}\frac{\partial}{\partial z}\left[S\frac{\partial S}{\partial z}\right]-\frac{Kg\propto(\rho_i-\rho_n)}{\delta_n}\frac{\partial S}{\partial z} \tag{18}$$

We now choose new dimensionless variables to convert equation (18) into dimension less form

$$T=\frac{K\beta}{L^2P\delta_n}t \ , \quad Z=\frac{z}{L}$$

$$\frac{\partial S}{\partial T}=S\frac{\partial^2 S}{\partial Z^2}+\left(\frac{\partial S}{\partial Z}\right)^2-A\frac{\partial S}{\partial Z} \tag{19}$$

$$where\ A=\frac{\propto gL(\rho_i-\rho_n)}{\beta} \ ,S(Z,T)=1-\propto S_i(Z,T)$$

This is governing equation of fingero-imbibition phenomenon in vertically downward direction and its solution represents saturation of injected fluid i.e. the average cross sectional area occupied by schematic fingers of average length at any depth Z at any time T.

The initial condition (15) is converted into

$$S(Z,0)=1-\propto S_0(Z) \tag{20}$$

The boundary conditions (16), (17) are converted into

$$S(0,T)=1-\propto S_{i0} \ , \qquad T>0 \tag{21}$$

$$S(L,T)=1-\propto S_{i1} \ , \qquad T>0 \tag{22}$$

4. SOLUTION BY GENERALISED SEPARABLE METHOD

In order to solve equation (19), we apply generalized separable method.

Let the generalised separable solution be of the form Polyanin and Zaistav [14]

$$S(Z,T)=P(Z)\emptyset(T)+Q(Z)\Psi(T)+R(Z)\tau(T), \tag{23}$$

As per Galaktionav and Posashkov [15], equation (23) can be expressed as

$$S(Z,T)=\emptyset(T)Z^2+\Psi(T)Z+\tau(T) \ , \tag{24}$$

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where the functions $\emptyset(T),\ \Psi(T)\ and\ \tau(T)$ are determined by a system of first order ordinary differential equations.

From equation (23),

$$\frac{\partial S}{\partial T}=\emptyset'(T)Z^2+\Psi'(T)Z+\tau'(T)$$

$$\frac{\partial S}{\partial Z}=2\emptyset(T)Z+\Psi(T)$$

$$\frac{\partial^2 S}{\partial Z^2}=2\emptyset(T)$$

Substituting the values of $\frac{\partial S}{\partial T},\frac{\partial S}{\partial Z},\ \frac{\partial^2 S}{\partial Z^2}$ in equation (19) and then equating the coefficients of different powers of Z, we get the system of first order ordinary differential equations with variable coefficients as follows:

$$\emptyset'(T)=6\emptyset^2\ \Rightarrow\ \emptyset(T)=-\frac{1}{6T+C_1} \tag{25}$$

$$\Psi'(T)=6\emptyset\Psi-2A\emptyset\Rightarrow\ \Psi(T)=\frac{C_2}{6T+C_1}+\frac{A}{3} \tag{26}$$

$$\tau'(T)=2\emptyset\tau\ +\ \Psi^{\,2}-A\Psi$$

$$\Rightarrow\tau(T)=\frac{-(C_2)^2}{4(6T+C_1)}+\frac{C_3}{(6T+C_1)^{1/3}}-\frac{AC_2}{3}-\frac{A^2(6T+C_1)}{18} \tag{27}$$

where C_1, C_2 *and* C_3 are constants of integration.

using the condition (19),

$$S(Z,0)=\emptyset(0)Z^2+\Psi(0)Z+\tau(0)$$

$$\Rightarrow 1-\propto S_0(Z)\ \ =\emptyset(0)Z^2+\Psi(0)Z+\tau(0) \tag{28}$$

where $\emptyset(0),\ \Psi(0),\ \tau(0)$ are non zero constants.

Let $\emptyset(0)=a\ ,\Psi(0)=b,\ \tau(0)=c$ where a, b and c are arbitrary constants.

Now to find C_1, C_2 *and* C_3, substituting the values of $\emptyset(0),\ \Psi(0)\ and\ \tau(0)$ in (25),(26) and (27) respectively, we get

$$C_1=-\frac{1}{a}\ ,C_2=\frac{1}{a}\left(\frac{A}{3}-b\right),C_3=\left(c+\frac{A^2}{36a}-\frac{Ab}{6a}-\right.$$

b24a−1a13

(29)

Substituting the values of C_1, C_2 *and* C_3 in (25),(26) and (27)

$$\emptyset(T)=-\frac{1}{6T-\frac{1}{a}} \tag{30}$$

$$\Psi(T)=\frac{\frac{1}{a}\left(\frac{A}{3}-b\right)}{6T-\frac{1}{a}}+\frac{A}{3} \tag{31}$$

$$\tau(T)=\frac{-\frac{1}{a^2}\left(\frac{A}{3}-b\right)^2}{4\left(6T-\frac{1}{a}\right)}+\frac{\left(c+\frac{A^2}{36a}-\frac{Ab}{6a}-\frac{b^2}{4a}\right)\left(-\frac{1}{a}\right)^{\frac{1}{3}}}{\left(6T-\frac{1}{a}\right)^{1/3}}-\frac{A}{3a}\left(\frac{A}{3}-b\right)-\frac{A^2\left(6T-\frac{1}{a}\right)}{18} \tag{32}$$

Substituting $\emptyset(T),\ \Psi(T)\ and\ \tau(T)$ in (24), we get

$$S(Z,T)=-\left(\frac{1}{6T-\frac{1}{a}}\right)Z^2+\left(\frac{\frac{1}{a}\left(\frac{A}{3}-b\right)}{6T-\frac{1}{a}}+\frac{A}{3}\right)Z+\left(\frac{-\frac{1}{a^2}\left(\frac{A}{3}-b\right)^2}{4\left(6T-\frac{1}{a}\right)}+\frac{\left(c+\frac{A^2}{36a}-\frac{Ab}{6a}-\frac{b^2}{4a}\right)\left(-\frac{1}{a}\right)^{\frac{1}{3}}}{\left(6T-\frac{1}{a}\right)^{1/3}}-\frac{A}{3a}\left(\frac{A}{3}-b\right)-\frac{A^2\left(6T-\frac{1}{a}\right)}{18}\right) \tag{33}$$

$$where\ A=\ \frac{\propto gL(\rho_i-\rho_n)}{\beta}$$

This is the required solution of governing equation (19). Numerical values and graphical representation of the solution (33) is obtained by Mat Lab coding as follows.

5. NUMERICAL AND GRAPHICAL PRESENTATION

Here for numerical calculation we consider the following values:

According to Meher et.al [8], the initial saturation of injected fluid is $S_0(Z)=e^{-Z}\ for\ any\ Z>0$.

Now substituting $S_0(Z)=e^{-Z}$ in equation (28), we get

$1-\propto e^{-Z}\ =aZ^2+bZ+c\quad where\ \propto=1.11.$

Using the expansion of e^{-Z} and equating the coefficients of Z^2, Z and constant term by neglecting Z^3 and higher powers of *Z*, we obtain the values of *a*, *b* and *c* as follows:

a $=-0.555$, b $=1.11$, c $=-0.11$

The values of some constants are taken from standard literature as follows:

$L=\ 1, g=9.8,\ \rho_n=0.3\ ,\ \rho_i=0.1,\ \beta=1\ ,\alpha=1.11$

$\Rightarrow A\approx 4.3$

Numerical and graphical presentations of solution (33) have been obtained by using MAT LAB coding. Figure (3) shows the graphs of S_i Vs. Z for time t = 0.5, 0.6, 0.7, 0.8 and Table-1 represent the numerical values.

TABLE I

SATURATION OF INJECTED WATER (S) FOR DIFFERENT Z FOR FIXED TIME T

Time →	0.5	0.6	0.7	0.8
Depth↓	**Saturation of injected water S(Z,T)**			
0	0.8559	0.8468	0.8378	0.8288
0.1	0.8514	0.8415	0.8317	0.8218
0.2	0.8382	0.8257	0.8131	0.8006
0.3	0.8161	0.7992	0.7822	0.7653
0.4	0.7852	0.7621	0.7390	0.7158
0.5	0.7455	0.7144	0.6833	0.6523
0.6	0.6969	0.6561	0.6154	0.5746
0.7	0.6395	0.5873	0.5350	0.4827

0.8	0.5733	0.5078	0.4423	0.3768
0.9	0.4983	0.4178	0.3372	0.2567
1	0.4144	0.3171	0.2198	0.1225

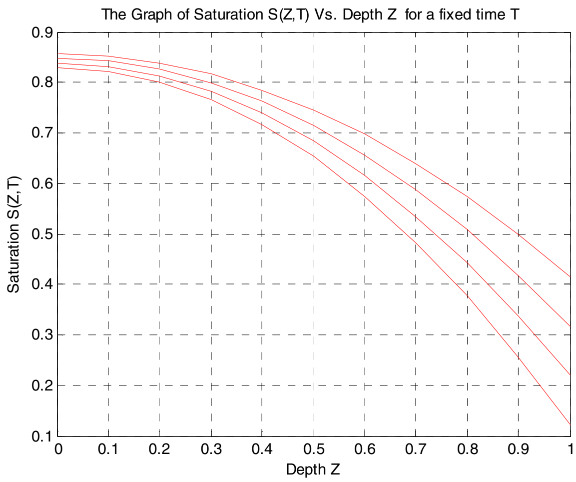

Figure: 3

The above graph shows that S(Z,T) is decreasing with respect to depth (Z) and time (T) but actually saturation of injected fluid is increasing with respect to depth as well as time which can be observed from figure (4).

Substituting $S(Z,T) = 1 - \propto S_i(Z,T)$ in (31),

$$S_i(Z,T) = \frac{1}{\alpha}\left[1 + \left(\frac{1}{6T - \frac{1}{a}}\right)Z^2 - \left(\frac{\frac{1}{a}\left(\frac{A}{3} - b\right)}{6T - \frac{1}{a}} + \frac{A}{3}\right)Z - \left(\frac{-\frac{1}{a^2}\left(\frac{A}{3} - b\right)^2}{4\left(6T - \frac{1}{a}\right)} + \frac{\left(c + \frac{A^2}{36a} - \frac{Ab}{6a} - \frac{b^2}{4a}\right)\left(-\frac{1}{a}\right)^{\frac{1}{3}}}{\left(6T - \frac{1}{a}\right)^{1/3}} - \frac{A}{3a}\left(\frac{A}{3} - b\right) - \frac{A^2\left(6T - \frac{1}{a}\right)}{18}\right)\right] \quad (34)$$

where $T = \dfrac{K\beta}{L^2 P\delta_n}t$, $\quad Z = \dfrac{z}{L}$ *and* $A = \dfrac{\propto gL(\rho_i - \rho_n)}{\beta}$

This is the required solution of governing equation (14) of the fingero imbibition phenomenon which represents the saturation of injected water occupied by the schematic fingers of average length at any depth Z for any time T > 0. Graphically, it can be represented as follows.

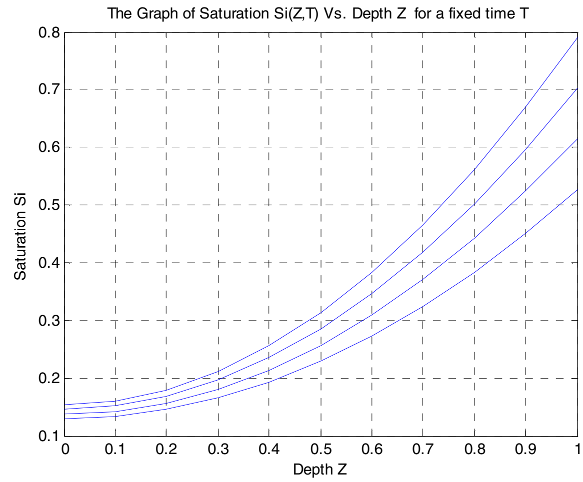

Figure: 4

6. INTERPRETATION AND CONCLUSION

The solution (33) represents Saturation of injected water in vertical downward direction for any depth Z for any time T > 0. The solution is in the form of quadratic polynomial in Z which is parabolic type and which satisfies the boundary conditions.

The graph of $S_i(Z,T)$ vs. Z for any time T > 0 is shown in figure (3) which shows that as the depth increases $S_i(Z,T)$ also increases for any time T > 0 which is consistent with physical phenomenon. In real, when depth is very high then gravitational effect play very less effect on saturation of injected fluid which shows by the graph that all family of curves are very close to each other. Also when injection rate is very high compare to gravitational effect then gravitational effect is negligible. In practical, many other parameters play important role in saturation of injected fluid but we have not considered those parameters for our particular interest.

For numerical values an Mat Lab coding, we have considered standard values of different parameters are taken from standard literature.

We can extend this work as future projection by adding other external effect and parameters which can increase the saturation of injected fluid.

7. REFERENCES

[1] Scheidegger, A.E. (1960), *The physics of flow through porous media*. University of Toronto Press, pp 224-229.

[2] Verma A.P.(1970), Water Resources Research(AGU), 6,906.

[3] Scheidegger AE and Johnson EF (1961). The statistical behaviour of instabilities in displacement process in porous media. Canadian J. Physics, 39 (2), pp. 326.

[4] Graham J. W. and Richardson J. G. (1959), Theory and application of imbibition phenomena in recovery of oil, Trans. AIME, 216.

[5] Mehta M.N. (1978), Composite expansion solution of finger-imbibition in double phase flow through porous medium, Proc.Indian Academic Science, vol-87A(2).

[6] Patel T.R and Mehta M. N. (2006), A Solution of the burger's equation arising into the imbibition phenomenon in double phase flow through porous media, International Journal of Physical Sciences, , 18 , 2, 255-260

[7] Yadav S.R and Mehta M.N. (2009), Mathematical model and similarity solution of Counter-current imbibition phenomenon in banded Porous matrix, Int. J. of Appl. Math and Mech., 5, 5, 76-86.

[8] Meher R and Mehta M.N. (2010). Adomain decomposition approach to fingero-imbibition phenomena in double phase flow through porous media, Int. J. of Appl. Math and Mech. 6 (9), pp. 34-36.

[9] Kinjal Patel and Mehta M.N. (2011), Power series solution of fingero- imbibition phenomenon in double phase flow through homogeneous porous media, *Int. J. of Appl. Math and Mech.* **7** (8): 65-77.

[10] Parikh et.al. (2013), Mathematical Model and Analysis of Counter-Current Imbibition in Vertical Downward Homogeneous Porous Media, British Journal of Mathematics & Computer Science, 3(4):478-489.

[11] Bear J. and Cheng A.H.D. (2009), Modeling Groundwater Flow and Contaminant Transport. Springer, Vol -23.

[12] Muskat (1946), The flow of homogeneous fluids through porous media, First edition, McGraw Hill Book Company.

[13] Mehta M.N. and Verma A.P. (1977). A Singular Perturbation Solution of Double Phase Flow due to differential wettability. *Indian J. Pure Appl. Math.* 8(5), pp 523- 526.

[14] Polyanin AD, Zaistav VF. Handbook of Nonlinear PDE, Chapmann & Hall/CRC; 2004.

[15] Galaktionov V. A., Posashkov, S. A (1989), Handbook of Nonlinear PDE, Chapmann & Hall/CRC.

Unified approach for solving initial valued and boundary valued Ordinary Differential Equations using wavelet collocation method

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***Abstract*--This article discusses approximation of solution for ordinary differential equation with wavelets. It presents a unified approach for wavelet collocation method; applied to solve both the initial valued problems as well as boundary valued problems. The method was found to give good agreement with the analytical solutions. We can directly solve the boundary valued problem as against the traditional shooting methods where the boundary valued problem itself is approximated by Initial valued problem and hence we achieve better accuracy with wavelet collocation methods.**

***Index Terms*--Boundary valued problem, initial valued problem, numerical solution of ordinary differential equation, wavelet collocation method.**

I. Introduction

OVER the last two decades wavelets are being effectively used for signal processing ,fingerprint verification[8], storing fingerprint electronically using wavelet, denoising data , musical tones, etc see [11] and solution of differential equations [2],[5], [3]. Wavelets have several properties which are encouraging their use for numerical solutions of differential equations. The orthogonal, compactly supported wavelet basis exactly approximates polynomial of increasingly higher order. This wavelet basis can provide an accurate and stable representation of differential operations even in region of strong gradients or oscillations. In addition, the orthogonal wavelet basis has the inherent advantage of multi resolution analysis over the traditional methods [3]. The adaptive wavelet collocation method is able to dynamically track the evolution of the solution's "irregular" features and to allocate higher grid density to the necessary regions. Therefore, the number of collocation points needed is optimized, without damaging the accuracy of the solution [2]. The benefit of Haar wavelet approach is their sparse matrices representation, fast transformation and possibility of implementation of fast algorithms [11].

In this paper we brief the collocation method used for solving initial valued problem and boundary valued problem with simple examples to establish a common unified approach of approximating derivative using wavelet function as basis. In this method higher order derivative is approximated using wavelet function and the lower order derivatives and functions itself are expressed by repeated integration. The orthogonal set of Haar functions is used. This group of square waves has magnitude ± 1 in some interval and zero elsewhere. These zeros make Haar transform faster than other square functions such as Walsh's functions. Haar wavelet basis lacks differentiability and hence the integration approach will be used instead of the differentiability for calculation of coefficients. Due to the local property of the powerful Haar wavelet the new method is simpler.

II. Haar wavelets

Haar wavelet transform has been used as an earliest example for orthonormal wavelet transform with compact support. The Haar wavelet transform is the first known wavelet and was proposed in 1909 by Alfred Haar.

The Haar wavelet family for, $x \in [0,1)$.

Is defined by

$$h_i(x) = \begin{cases} 1 & x \in [\alpha, \beta) \\ -1 & x \in [\beta, \gamma) \\ 0 & elsewhere \end{cases} \qquad (1)$$

with

$$\alpha = \frac{k}{m} \quad \beta = \frac{k+0.5}{m} \quad \gamma = \frac{k+1}{m} \quad (2)$$

here m= 2^j , j=0, 1, …, J. indicates the levels of wavelet and integer k=0,1,..,(m-1), the shift parameter. Maximum resolution is J, and i=m+k+1. Incase of minimum value m=1, k=0, i=2.

The maximal value of i is i=2M=2J+1.

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The function

$$h_1(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & elsewhere. \end{cases} \qquad (3)$$

From equation (1),

$$h_2(x) = \begin{cases} 1 & x \in [0, 0.5) \\ -1 & x \in [0.5, 1) \\ 0 & elsewhere. \end{cases}$$ can be graphically visualized as in fig 1.

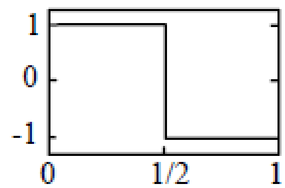

Fig 1, Haar function.

The equation (3) is also called mother wavelet. In order to perform wavelet transform, Haar wavelet uses translations and dilations of the function, i.e. the transformation uses the following function

$$h(x) = h(2^j x - k) \qquad (4)$$

Translation / shifting $h(x) = h(x-k)$

Dilation / scaling $h(x) = h(2^j x)$

where this is the basic work for wavelet expansion.

With the dilation and translation process as in Eq.(4), one can easily obtain father wavelet, daughter wavelet, granddaughter wavelet etc.

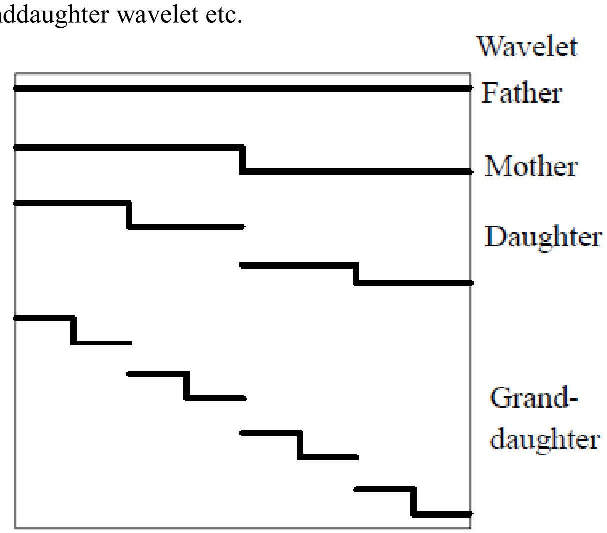

Fig 2 represents Haar wavelets up to two resolutions

We can obtain coefficient matrix H of order $2m \times 2m$ as

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$ for m=2.

The Haar wavelets are orthogonal as, i.e.

$$\int_0^1 h_i(x) h_l(x) dx = \begin{cases} \frac{1}{m} & for \quad i = l \\ 0 & for \quad i \neq l \end{cases} \qquad (5)$$

The operational matrix P which is a 2m square matrix is define as in [12] by

$$p_{i,1}(x) = \int_0^x h_i(x') dx' \qquad (6)$$

and the recurrence relation is given by

$$p_{i,v+1}(x) = \int_0^x p_{i,v}(x') dx' \quad where\ v = 1, 2, \ ... \qquad (7)$$

We will need the integral

$$P(x) = \underbrace{\int_A^x \int_A^x ... \int_A^x h_i(t) dt^u}_{\text{u-times}} = \frac{1}{(u-1)!} \int_A^x (x-t)^{u-1} h_i(t) dt \qquad (8)$$

with $u = 2, 3, ..., n \quad and \quad i = 1, 2, ..., 2m.$

The above integrals can be evaluated using equation (1); the first two are given by

$$p_{i,1}(x) = \begin{cases} x - \alpha & for \quad x \in [\alpha, \beta), \\ \gamma - x & for \quad x \in [\beta, \gamma), \\ 0 & elsewhere. \end{cases} \qquad (9)$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2 & for \quad x \in [\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & for \quad x \in [\beta, \gamma), \\ \frac{1}{4m^2} & for \quad x \in [\gamma, 1), \\ 0 & elsewhere \end{cases} \qquad (10)$$

and so on will be utilized in representing the derivative and function values in the further discussion.

III. FUNCTION APPROXIMATION.

Considering the fact that Haar functions are orthogonal, we may take any function $f(x)$ which is square integrable in the interval $[0,1]$ as an infinite sum of Haar wavelets,

$$f(x)=\sum_{i=1}^{\infty}a_i h_i(x)$$

where a_i are haar coefficients and $h_i(x)$ are haar wavelet functions.

$f(x)$ has finite terms if $f(x)$ is piecewise constant or can be approximated as piecewise constant during each subinterval as,

$$f(x)=\sum_{i=1}^{2M}a_i h_i(x) \qquad (11)$$

In this scheme the highest order derivative of the function is approximated by haar wavelets and the consecutive lower order derivatives and function itself is obtained by repeated integration as explained in the section IV and V.

IV. SOLVING INITIAL VALUED PROBLEMS.

Consider the general nth order linear differential equation

$$M_1 y^n(x)+M_2 y^{(n-1)}(x)+\ldots+M_n y(x)=f \qquad (12)$$

$for \quad x\in[A,B]$ with initial conditions

$y^{(n-1)}(A), \quad y^{(n-2)}(A), \ldots, y(A)$ are known.

We assume that $y^n(x)=\sum_{i=1}^{2M}a_i h_i(x) \qquad (13)$

and all lower order derivative approximations are obtained by repeated integrals. For example r[th] order derivative of y is obtained as

$$y^r(x)=\sum_{i=1}^{2M}a_i p_{i,n-r}(x)+\sum_{\sigma=0}^{n-r-1}\frac{1}{\sigma!}(x-A)^{\sigma}y_0^{(r+\sigma)} \qquad (14)$$

We obtain $y^{(n-1)}(x),\ y^{(n-2)}(x),\ldots \quad and \quad y(x)$.

The collocation points are obtained by

$$x_p=\frac{(p-\frac{1}{2})}{2M}, \quad p=1,2,\ldots 2M. \qquad (15)$$

The expressions of $y^n(x),\ y^{(n-1)}(x), \quad \ldots and \quad y(x)$ are substituted in differential equation, discretization is applied along the points given by equation (15) resulting in a linear or non linear system of $2M\times 2M$. Solving the system for haar coefficients the approximate solution is achieved.

V. SOLVING BOUNDARY VALUED PROBLEM.

Consider a second order boundary valued problem

$$y''(x)=f(x,y,y') \qquad for \quad x\in[0,1] \qquad (16)$$

For second order ordinary differential equations, there are four different types of boundary conditions possible. They are treated differently as follows;

a) $y(0)=R \quad and \quad y(1)=Q$, then integrating

equation (16) yields $y'(x)=\sum_{i=1}^{2M}a_i p_{i,1}(x)+y'(0)$

as $p_{i,1}(0)=0$.

Integrating again and using condition $y(0)=R$ we

get $y(x)=R+y'(0)x+\sum_{i=1}^{2M}a_i p_{i,2}(x)$.

Now utilizing second condition $y(1)=Q$, we obtain

$$y'(0)=(Q-R)-\sum_{i=1}^{2M}a_i c_{i1}$$

$with \quad c_{i1}=\int_0^1 p_{i,1}(x)dx$ further simplifying we get

$$y(x)=R+(Q-R)x+\sum_{i=1}^{2M}a_i(p_{i,2}(x)-xc_{i1}) \qquad (17)$$

and

$$y'(x)=Q-R+\sum_{i=1}^{2M}a_i(p_{i,1}(x)-c_{i1}) \qquad (18)$$

b) $y'(0)=R_1 \quad and \quad y(1)=Q_1$

Integrating equation (16) and using boundary condition $y'(0)=R_1$, we get

$$y'(x)=R_1+\sum_{i=1}^{2M}a_i p_{i,1}(x) \qquad (19)$$

and

$$y(x)=Q_1-R_1(1-x)-\sum_{i=1}^{2M}a_i(c_{i1}-p_{i,2}(x)) \qquad (20)$$

c) $y(0)=R_2 \quad and \quad y'(1)=Q_2$

we get $y'(x)=Q_2-a_1+\sum_{i=1}^{2M}a_i p_{i,1}(x) \qquad (21)$

and

$$y(x)=R_2+(Q_2-a_1)x+\sum_{i=1}^{2M}a_i(p_{i,2}(x)) \qquad (22)$$

d) $y'(0)=R_3 \quad and \quad y'(1)=Q_3$

here integrating equation (16) we get by implementing first condition

$$y'(x) = R_3 + \sum_{i=1}^{2M} a_i p_{i,1}(x) \quad using \;\; y'(1)$$

$$(Q_3 - R_3) = a_1 \quad as \quad p_{i,1}(1) = 1.$$

so

$$y''(x) = (Q_3 - R_3)h_1(x) + \sum_{i=2}^{2M} a_i h_i(x) \qquad (23)$$

$$y'(x) = R_3 + (Q_3 - R_3)p_{1,1}(x) + \sum_{i=2}^{2M} a_i p_{i,1}(x) \qquad (24)$$

$$y(x) = y(0) + R_3 x + (Q_3 - R_3)p_{1,2}(x) + \sum_{i=2}^{2M} a_i p_{i,2}(x) \qquad (25)$$

which is obtained by equation (24), by integrating from 0 to x. Similarly we can extend the approach to higher order differential equations with boundary conditions.

Section VI discusses the unified algorithm for implementation of the wavelet collocation method, for both initial valued problems and boundary valued problem.

VI. ALGORITHMS

i) The highest order derivative is approximated by haar wavelet function.

ii) The successive lower order derivatives and the function itself is replaced by the expressions obtained by repeated integration obtained in (i).

iii) The algebraic expression in terms of haar coefficients is represented in matrix form.

iv) The matrix is solved to obtain the haar coefficients a_i's which are then substituted in the expression of solution function.

Separate MATLAB routines are generated for computation of the matrix P and C which appear in the algebraic representation. P represents the matrix formed by $p_{i,k}$'s and C represents the matrix formed by required c_{ik}'s, where k depends on the order of the equation handled.

Now to get a clear idea of the methods we give examples, one each for second order initial valued problem and second order boundary valued problem in section VII and VIII along with the plot of haar solution compared with the exact analytical solution for the same.

VII. EXAMPLE 1.

Consider an initial valued ordinary differential equation

$$y'' + y = \sin x + x\cos x \qquad x \in [0,1]$$

$$and \quad y(0) = 1,\; y'(0) = 1.$$

The analytical solution is

$$y(x) = \cos x + \frac{5}{4}\sin x + \frac{1}{4}(x^2 \sin x - x\cos x)$$

Wavelet formulation is obtained by substituting

$y''(x) = \sum_{i=1}^{2M} a_i h_i(x)$ and integrating twice this equation we get

$$y(x) = \sum_{i=1}^{2M} a_i p_{i2}(x) + 1 + x$$

The differential equation gets converted to

$$\sum_{i=1}^{2M} a_i (h_i(x) + p_{i,2}(x)) = \sin x + x\cos x - 1 - x.$$

Solving the system $A[H+P] = B$ we obtain the wavelet coefficient matrix A, where H is the haar matrix, P is the matrix consisting with rows $p_{i,k}$'s and B is right hand side vector obtained by considering values of x at collocation points. From the expression of $y(x)$ by substituting the wavelet coefficients, the solution function is generated.

For j=3, that is m=16. We obtained the result, compared it with analytical solution using MATLAB program.

The plot is given in fig 3.

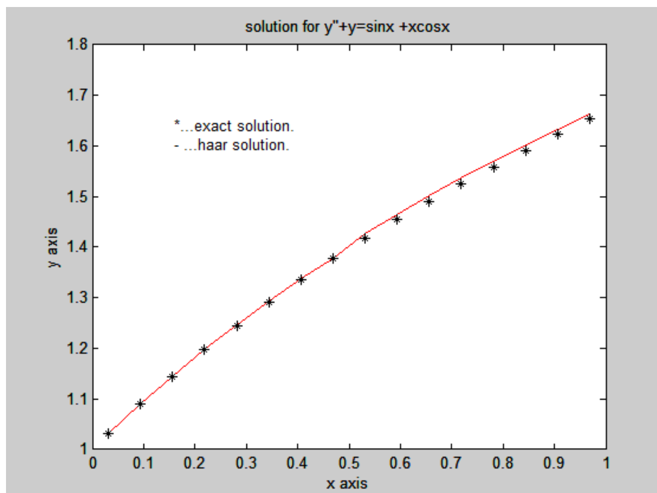

Fig 3, represents comparison of haar solution with exact solution for an initial valued ordinary differential equation for j=3.

Now fig 4 represents the graph of the haar, analytical and inbuilt function implementation of matlab results for the initial valued problem (example 1)

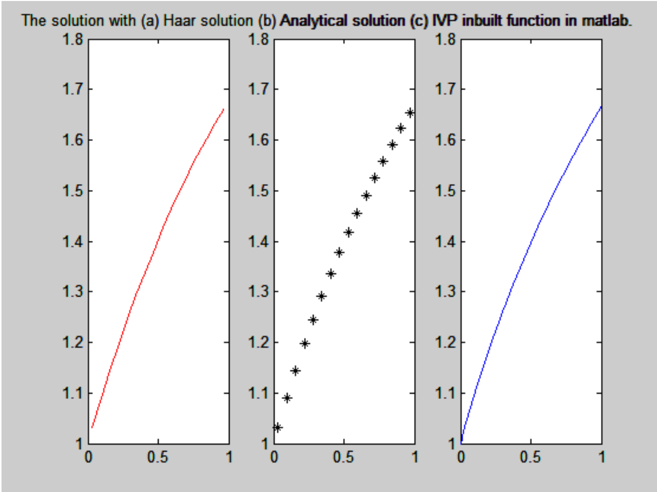

Fig 4.

VIII. Example 2.

Consider a second order boundary valued problem as

$$y'' = y' + y + e^{x}(1-2x) \qquad x \in [0,1].$$

$$with \quad conditions \quad y(0)=1, \quad y(1)=3e.$$

which are the boundary conditions as mentioned in type (a) in section V

The analytical solution of this boundary valued problem is $y = e^{x}(1+2x)$, the wavelet formulation as per section V (a) we obtain

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

$$y'(x) = 3e - 1 + \sum_{i=1}^{2M} a_i (p_{i,1}(x) - c_{i1}) \quad and$$

$$y(x) = 1 + (3e-1)x + \sum_{i=1}^{2M} a_i (p_{i,2}(x) - x c_{i1})$$

Therefore the boundary valued problem gets transformed as

$$\sum_{i=1}^{2M} a_i (h_i(x) - p_{i,1}(x) + c_{i1}(1+x) - p_{i,2}(x)) = x(3e-1)$$
$$+ e^{x}(1-2x) + 3e.$$

For 2m collocation points we get 2m linear algebraic equations with unknowns 2m haar coefficients, which are obtained by solving the system of equations. By substituting the coefficients a_i so obtained in the equation for $y(x)$ we get the required solution.

This result is compared with the analytical solution which is exactly similar at j=3, that is m=16, as in fig 5.

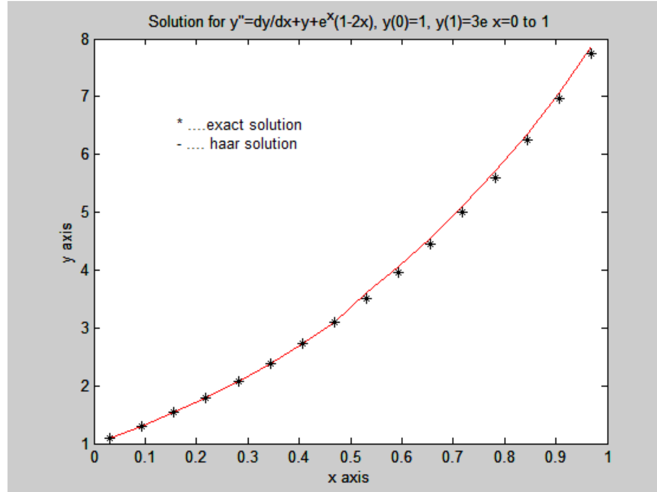

Fig 5, comparison of haar solution with exact solution for boundary valued ordinary differential equation for, j=3.

Now fig 6 represents the graph of the haar, analytical and inbuilt function implementation of matlab results for the boundary valued problem (example 2).

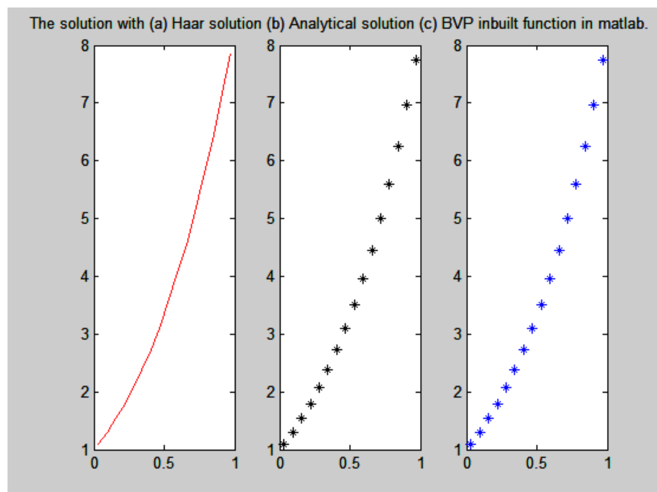

Fig 6.

IX. Conclusion

We have presented a unified way for solving the initial valued problem and the boundary valued problem using the wavelet collocation method. As we stated earlier this method is bound to give more accurate solution for boundary valued problem compared to the traditional shooting methods. Also an appropriate higher wavelet resolution may be chosen if the ordinary differential equation is stiff. To improve the accuracy and optimize the computation an appropriate dynamic resolution adaptive scheme could be formulated.

In case of nonlinear differential equation with relatively less non linearity, it generates manageable non linear algebraic equations; otherwise it becomes a complicated system. Our method is more amicable for linear differential equations.

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X. References

[1] C.F.Chen, C.H.Hsiao. Haar wavelet method for solving lumped and distributed parameter systems. IEE Proce.- Control Theory and Appl., vol 144,N0.1, 1997.

[2] Paulo Cruz, AdeHlio Mendes, F.D. Magalhaes. Using wavelets for solving PDEs: an adaptive collocation method. Chemical Engineering Science 56 (2001)

[3] M.A.H. Dempster, A. Eswaran. Solution of PDE by Wavelet Methods. Research Papers in Management Studies. WP/2001.

[4] R.J.E. Merry. Wavelet Theory and Applications. A literature study. Eindhoven, June 7, 2005.

[5] Haydar Akca, Mohammed H. Al-Lail, Val´ery Covachev. Survey on Wavelet Transform and Application in ODE and Wavelet Networks Advances in Dynamical Systems and Applications. ISSN 0973-5321 Volume 1 Number 2 (2006), pp. 129–162.

[6] Zhi shi, Li-Yuan Deng, Qing- Jiang Chen. Numerical solution of differential equation by using Haar wavelet. Proceedings of 2007 international conference of Wavelet Analysis and Pattern Recognition, Beijing China, Nov. 2007.

[7] Tonghua Zhang, Yu-Chu Tian and Moses O.Tade. Wavelet based Collocation Method for Stiff Systems in process engineering. Journal of Mathematical chemistry, Vol 44, No.2, pp. 501-513, 2008.

[8] M. Sifuzzaman, M.R. Islam and M.Z. Ali. Application of Wavelet Transform and its Advantages Compared to Fourier Transform. Journal of Physical Sciences, Vol. 13, 2009, 121-134.

[9] Mani Mehra. Wavelet and Differential Equations-a short review. 01.30.Cc. http://web.iitd.ac.in/~mmehra, May 2009.

[10] Siraj‐ul‐Islam, B. Sarler, I. Aziz and F. Haq "Haar wavelet collocation method for the numerical solution of boundary layer fluid flow problems" International Journal of Thermal Sciences, 52 (2011).

[11] Manoj Kumar, Sapna Pandit. Wavelet Transform and Wavelet Based Numerical Methods: an Introduction International Journal of Nonlinear Sciences (2012).

[12] V.Mishra, H.Kaur, R.C.Mittal. Haar wavelet Algorithm for solving certain differential, integral and integro differential equations .Int.J. of appl. Math and Mech. 8(6) 2012.

[13] Adefemi sunmonu. Implementation of wavelet Solutions to Second order differential equations with Maple .Applied Mathematics and Science, Vol 6, 2012.

Imparting Communication Skills in Technical Courses: A Paradigm Shift

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Abstract--**A course in communication skills in technical courses usually receives minimal attention from students and teachers alike. However its importance is realized only when students are required to compile practical research or experiments into a written assignment for submission, or a presentation, proposal for research, dissertation or a thesis. This may occur at both the under graduate as well as post-graduate courses of technical education. This inadequacy usually results in poor technical writing, plagiarized reports and ill organized essays in written communication as well as substandard performance in recruitment interviews, lack of confidence in presentations and ill-organized meetings in terms of oral communication. In order to avoid this, the students should be made aware of the skills that they are required to practice by acquainting them in advance about the possible repercussions of ignoring the components and the course requirements of technical communication skills. This paper explores some of the issues concerning the necessity of communication skills in technical education, which components should receive prime focus, how these components can be best imparted, the expectations from the students, the role of the instructor and reflections upon how the course can pronounce a positive impact upon the students in the long run.**

***Index Terms*--, Communication Skills, engineering , English, Technical courses.**

A course in communication skills in technical education usually receives peripheral attention from students and teachers alike. However its importance is realized only when students are required to compile practical research or experiments into a written assignment for submission, or a presentation, proposal for research, dissertation or a thesis. This may occur at both the under graduate as well as post-graduate courses of technical education. This inadequacy usually results in poor technical writing, plagiarised reports and ill organized essays in written communication as well as substandard performance in recruitment interviews, lack of confidence in presentations and ill-organized meetings. In order to avoid this, the students should be made aware of the skills that they are required to practice by acquainting them in advance about the possible repercussions of ignoring the components and the course requirements of technical communication skills. This paper explores some of the issues concerning the necessity of communication skills in technical education, which components should receive prime focus, how these components can be best imparted, the expectations from the students, the role of the instructor and reflections upon how the course can pronounce a positive impact upon the students in the long run.

Most schools in India recommended the learning of English language as a subject in the syllabus, recognizing the fact that it has attained the position of an associate official language in our country. Likewise, in most undergraduate syllabi particularly in technical education in engineering colleges this subject further features as the study of communication skills. Although there is no written rule which endorses that the study of Communication Skills in technical education should be in English, it is usually taught and learnt in this language in acknowledging its stature as the global language of the world. In most cases the content of the communication skills course is designed to help students to communicate well usually with the existent knowledge of English language. Alternatively it serves as a remedial course, enabling the student to unlearn what has been wrongly learnt or learning all that has been missed to be learnt in school. Since it is a basic course that supports students throughout their career as a student and a professional it is usually studied by the students in the first or second semester of the curriculum.

At this juncture it is important to briefly outline what are communication skills. Having its origins in the Latin word “Communis” meaning to share the word communication includes activities such as the sharing of ideas, suggestions, information and so on through listening, speaking, reading and writing. The process of communication comprises primarily of sending and receiving messages. When this message is clear and effective, there is little scope of ambiguity and misunderstanding. Recent reports have indicated that very few candidates in proportion to those who appear for interviews are identified with employability skills. They may possess sound technical skills but they lag behind the others in communication skills. Hence the study, practice and application of communication skills have become indispensible to the technical professionals as well.

In most universities, courses in business and technical communication have grown, though slowly and unevenly. The growth has been fuelled by the demands of industry for graduates who could write and by requirements from accrediting institutions for instruction in communication used at the workplace. It is currently nurtured by the

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burgeoning research base which led to tenured positions for atleast some of the faculty teaching these courses.

From a student's perspective there are various issues concerning the necessity of communication skills in technical education. Technical communication is an indispensable course for all students and professionals. The advent of new technologies has caused the world to shrink into a global village. As a result, effective communication has become essential. While the students experience the need for successful oral and written communication in their academic activities, professionals have to tackle the challenges of communication effectively and efficiently at their workplace.

An ideal course in technical communication should enable the learner to encounter the issues in communication primarily in a technical milieu as communicating formal technical messages is vital both for students and professionals. This can be accomplished by underscoring the significance of both oral and written communication in myriad circumstances. Learners may be encouraged to devise certain techniques to improve confidence and effectiveness both for making presentations and for working as a member of a team.

One of the prime challenges that a student is faced in technical education in the rationing of time among laboratory work, key components of his or her course and refining communication skills. It is needless to mention, that the third component receives very little or no attention.

A course in communication skills in technical education is commonly taught in under-graduate engineering, Computer Applications, Science, and in the PG level it can exist all- pervasively as a supplementary course for remedial teaching depending upon the communication competence of the students.

An integral component of communication skills should be to develop some of the key skills of students like refining listening, questioning and explaining skills as well as presenting or defending a position clearly. Hence certain components of communication should receive prime focus. These include the basic theory of communication including process, barriers, forms and methods of communication, technology in communication, listening, group communication like effective presentation strategies, interview skills, meetings and conferences, paragraph development, précis writing and reading comprehension. Written forms of communication could include résumé, formal letters, memos, emails, technical reports, technical proposals, research papers, dissertations, theses instruction manuals and technical description.

There can be many ways in which these components may be best imparted. Communication skills can be taught particularly in the undergraduate courses by highlighting the uses of effective communication. Gathering evidence relating to written communication is comparatively easy and can be incorporated into most areas of the curriculum. Evidence of oral communication skills can be built into many parts of the curriculum in most subject areas, such as those associated with individual student or group presentations. Such evidence often arises from activities that represent good opportunities for learning, and for students to gain feedback not only from their lectures but from each other.

Today Information and Communication Technologies play an important role in enhancing quality of education. ICTs (Information and Communication Technologies) are a diverse set of technological tools and resources used to communicate, and to create, disseminate, store, and manage information. These technologies include computers, the Internet, broadcasting technologies (radio and television), and telephones. According to Anderson the growth in the use of computer based learning resource materials assists students to collect evidence of their use of information technology, and to do so in the context of their subject based studies (in this case, communication skills) rather than having to undertake additional tasks to accumulate this type of evidence. When students use information technology for communication, such as email, they simultaneously reveal their proficiency in their communication skills. Additional use of information technology may involve students' activities with the Internet, emailing assignments, using spreadsheets or desktop publishing packages in the context of their coursework or class-work. (21)

Since the new ICTs are assisting in achieving many objectives of education particularly with cooperative and collaborative learning approach (of constructivist paradigm), there is a need to assess the effectiveness of ICTs in education. This could be done through planned research studies. We need to take a fresh look at the research methodologies which we have been using in Education for the new ICTs as they demand a new perspective. The use of ICT in language teaching can implemented with the appropriate use of Google groups, blogs, Slideshares and so on.

Analytical thinking and preparing persuasive messages can be taught though innovative ways. For example, a simple exercise of printed and electronic advertisements of popular products can be conducted. Slogans of well-known products can be hung up on the walls of a class room, and the students could be asked to identify the products and critically assess the use of these punch lines. All the tasks can be aimed at understanding analytical thinking and developing it. Thus the teachers could creatively use different materials and methods to evoke critical thinking in their ELT classrooms.

Conducting well organised meetings can be taught through simulation. Some students could be engaged in preparing the agenda, others could arrange for the necessary infrastructure. Issues, though imaginary could be discussed and some learners may be asked to record the proceeding of the meeting by noting down the minutes of the meeting. This gives the participants the necessary practice in both oral as well as written communication. Similar exercises may be conducted for teaching interviews. A group of students may become the panel of interviewers while other students may assume of the role of interviewees. The role of the instructor becomes that of a facilitator and intervenes only to give relevant inputs and feedback to the students' performances. Such role-playing has been found to be popular among both graduate and undergraduate students. Brief and snappy demonstrations of a presentation on a general or technical topic by the instructor can guide students about how to deliver an effective presentation and what pitfalls to avoid while doing so. However if the instructor takes too long "explaining the 'right' way of doing something (then) pupils who are itching to try it themselves at first become turned off". (Brown et al 15) Further the teacher should convey to the participants that "Getting other people to do what we want them to do is the one of the most valuable skills in the workplace." (Denny 34) However one should learn to motivate rather than manipulate in order to make such communication effective. Participants should remember, "The difference between good leadership and poor management is the difference between motivation and manipulation." In order to motivate, one should have good communication with an individual and have high emotional intelligence that is characterised by self-awareness, self-regulation, motivation, empathy and social skill. Finer points like those enumerated below may also be discussed or brainstormed in the classroom like:

- a word or two of praise maintains enthusiasm and good performance at the workplace.
- Criticism is only acceptable if it is constructive and leads to positive communication that will eliminate errors and enhance performance. (Ibid. 35-36)

While teaching letter writing, the teacher should clarify how the "physical preparation of letter writing is far from complex business". (Adair 104) This can be accomplished by making the learners collect relevant facts and materials together, reading any related previous correspondence in order to get a clear picture and finally sifting through the facts to sort out the relevant information to be included in the first draft of the letter. (Idem.)

Various studies in English as a Foreign Language/English as a Second Language (EFL/ESL) contexts have confirmed the positive correlation between analytical thinking and reading comprehension. Reading comprehension frequently demands logical thinking and is thus an integral aspect of technical communication. To this end, all question types associated with general and Critical Reading Questions, Vocabulary in Context, Literal Comprehension, and Extended Reasoning can be identified. This can strongly influence students' attention when reading different passages. Given that learning a foreign language, specially at intermediate and advanced levels, calls for a good deal of flexibility and the deployment of higher order thinking skills, analytical thinking can be seen as a contributory factor to the success of foreign language learners and students in reading comprehension. Students can be encouraged to read and summarise or answer questions based on technical reports or news items for practice.

Certain demographics within the user population (older adults, disabled persons, non-native speakers) may face serious challenges while attempting to use self-service documentation. Technical communication educators should prepare students to function as user advocates for members of these groups. Technical communication students need a thorough understanding of the challenges that may interfere with an audience's ability to use websites and other online documentation. Web writing may include blogs, news updates, and even email messages. In order to give letter and email writing practice to students they may be given real life situations and asked to exchange formal letters or emails with one another. This encourages peer learning and students can give a feedback for the messages they receive and learn from the mistakes that they or other learners may commit. This induces a healthy learning environment with minimal intervention of the instructor. Resume is a vital form of communication for selling one's self to the recruiter. Preparing a CV and a cover letter requires diligence and practice and although there are general directives for both, there is no universal yardstick. Here the learners may be made to realize how customizing their resume and covering letter according to the recruiters' requirements can help achieve the desired results. A detailed demonstration of how to acknowledge sources or references and give citations to avoid plagiarism is greatly helpful in preparing assignments, proposals and dissertations. Hence this can be elaborately shown in the class.

The expectations of the students are largely tangible outcomes at the end of the course. In alignment with their individual learning outcomes, they hope to be imparted some practical guidelines which they can use in their various forms of oral and written communication in both formal and informal contexts. It is expected from the students that each session module and course component along with its assignment should be duly completed through all fair means by every student to achieve this end.

The role of the instructor in teaching communication skills is challenging and diverse. In an age where education is the cause of concern for the youngest country in the world, fundamental skills such as language skills, if they are to be imparted to our generation next, should be offered at the basic levels of a student's education even before they complete their school education. However wherever this is not accomplished, it is in educational institutions offering undergraduate and post graduate courses that this should be achieved in order to enhance a student's employability skills, so that his/her transition from being a student to a professional is a smooth process.

Students should be sensitized to a unique learning process which continues to pay dividends throughout their lives. The focus of such a course should be to develop conceptual understanding via Active learning, interactive classes, inculcate the habit of self learning, pre-class reading and develop problem solving skills. Lectures in interactive mode through upload of study material on university website

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can persuade students to come prepared for a session. Students could be asked concept related pre-planned questions, generate group discussion, focus on peer learning (question pairs) and quizzes and oral tests.

Since most of the students who might study communication skills as a part of their curriculum are adults, it is advisable to justify the contents of the course in order to trigger off an andragogical approach in the classroom. Students work best when they are aware of what they are trying to do and why they are made to study certain components among others. Thus the relevance of the course contents should be clear to them. In addition, Anderson and others have pointed out that the intended leaning outcome of each session should be articulated to them with the explanation regarding why it is worthwhile for students to strive to achieve them. (27) As a result of this the students can gain a self-direction in learning. This further helps students to progressively take the responsibility for their grades and their learning. For instance they should be able to clarify their goals as learners.

It has been observed that "the key strategies of communication should be substantiated to the learners with examples from real-life workplace situations. By amalgamating theory and practice the instructor should be able to bring forth real situations of communication in the discussion." (Raman, Sharma iii) In written communication the kinds of content, writing style, text organization and format for various types of technical documents should be elaborately discussed. They should address the students with real-life examples and orienting the strategies to practical application.

An instructor should attend to what takes place by observing and listening to students' involvement and contributions in terms of class participation. They should acquaint students with the consequences of participating, in a group discussion for example. In addition students should be given realistic and achievable targets. For instance, students could be asked to prepare for a ten-minute presentation on a general topic at the notice of one week. Through various activities the teacher could give students the understanding of personal growth, communication skills, group and team work skills and self-direction in learning. The instructor should demonstrate to the students the notion of an academic discourse or intellectual process and academic arguments in their discipline. For instance, he/she can show why certain words and abbreviations in a particular kind of writing are barred from formal academic writing. Another example could be that of a group discussion which could help students to develop their critical and analytical thinking. It can also help them to review evidence in the light of the theories. Grouping students for group activities can enable students to learn to collaborate and work as an effective group or a team.

Reflections upon how the course can pronounce a positive impact upon the students in the long run can be done by the instructor periodically in various ways. Students' self-reflection and the instructor's self-introspection can jointly achieve this purpose. In order to improve technical communication skills learning language registers can offer an extra edge. For example, students pursuing a Bachelor of Computer Applications may learn computer application oriented vocabulary and related jargons which they can relate to the other courses in their syllabus.

Assessment of students can be formative based on continuous evaluation rather than summative alone—based on a test at the end of the course. Depending upon the degree of autonomy that is granted to the tutor, interactive question papers or test papers maybe generated instead of theory oriented questions. For instance, instead of asking a question like "What are the different layouts of a business letter and explain each of them in detail." An illustration of a letter may be given with the question "Identify the layout of the letter and state whether it is a good or a bad example of a letter justifying your answer in detail." This also gives the instructor continuous feedback about the performance of the students and the instructor himself/herself.

For best results such a course may be tailor-made according to the requirements of the students based on their performance in some accurate diagnostic tests on communication skills in the beginning of the course. It should be remembered that "Using language effectively, particularly in business and professional settings, is not always easy. The relationship between a word and what it represents is not always based on real or concrete shared characteristics that can be analyzed or predicted." (O'Hair 145) One can increase one's skill in using language by continually learning new ways to express what one means. The changing world of turbulent times has witnessed the need not only for managers but also technical professionals to embody the skills of effective communication. Effective communication will help increase the commitment and motivation of the employees as a part of a workforce and is the direct reflection of one's behaviour. Without good communication, there is less involvement, resistance to change and the risk of misunderstanding. This ensures that the participants become "industry ready", effective communicators with high employability skills, good leaders and thereby ideal and desirable employees in the work-place.

I. References

Adair, John. *The Effective Communicator*. Mumbai: Jaico Publishing House, 1995, 2002.

Anderson, David, Sally Brown and Phil Race. *500 Tips for Further and Continuing Educating Lecturers*. London: Kogan Page, 1997

Brown, Sally, Carolyn Earlyn and Phil Race. *500 Tips for Teachers*. London: Kogan Page, 1994, rpt.1998.

Denny, Richard. *Communicate to Win*. New Delhi: Kogan Page, Second Edition, 2008.

O'Hair, Dan, Gustav W. Fredrich, D. D. Dixon. *Strategic Communication in Business and Professions.* New Delhi: Pearson Education. 2008.

Raman, Meenakshi and Sangeeta Sharma. "Preface." *Technical Communication: Principles and Practice*. New Delhi: Oxford University Press, 2004.

978-1-4673-1719-1/12/$31.00©2013IEEE

Application of Variational Homotopy Perturbation Method in Oil Recovery Process

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***Abstract*-- In this paper an effort is made to present theoretical model for the oil recovery problem in porous media for the two phase flow.The Analytical Solution and graphical illustration of the oil recovery problem is presented by means of Variational Homotopy Perturbation method which is formulated by the coupling of variational iteration method and Homotopy Perturbation Method. The graphical representation is givenby using MATLAB coding.**

***Index Terms*-- Variational Homotopy Perturbation method, porous media, oil recovery.**

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I. INTRODUCTION

This paper is devoted to the study of mathematical modeling of oil and gas recovery This work is devoted to the mathematical modeling of water-flooding, which is the most important and widely used method of oil recovery. Buckley-Leverett equation, which describes two-phase displacement of immiscible liquids through porous media, and the simultaneous flow of oil and water, in particular. Water-flooding of reservoirs, which is the basic process involved in the development of oil fields. He [5-10] developed the variational iteration and homotopy perturbation methods for solving linear, nonlinear, initial and boundary value problems. In these methods the solution is given in an infinite series usually converging to an accurate solution. In a later work Ghorbani et al. [11] split the nonlinear term into a series of polynomials calling them as the He's polynomials. Most recently, Noor and Mohyud-Din[12],[13] used this concept for solving nonlinear boundary value problems. The basic motivation of this paper is the introduction of the Variational Homotopy Perturbation Method (VHPM) which is formulated by the coupling of variational iteration method and He's polynomials. The proposed VHPM provides the solution in a rapid convergent series which may lead the solution to a closed form. In this technique, the correction functional is developed [7],[9],[10] and the Lagrange multipliers are calculated optimally via variational theory. The use of Lagrange multipliers reduces the successive application of the integral operator and the cumbersome of huge computational work while still maintaining a very high level of accuracy. Finally, He's polynomials are introduced in the correction functional and the comparison of like powers of p gives solutions of various orders. The proposed iterative scheme takes full advantage of variational iteration and the homotopy perturbation methods and absorbs all the positive features of the coupled techniques.

II VARIATIONAL HOMOTOPY PERTURBATION METHOD

To convey the basic idea of the Variational homotopy perturbation method, we consider the following general differential equation

$$Lu + Nu = g(x) \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and g(x) is the forcing term. According to variational iteration method , we can construct a correct functional [9],[10] as follows:

$$u_{n+1}(x) = u_{n(x)} + \int_0^t \lambda(\tau)\left(Lu_n(\tau) + Nu_n(\tau) - g(\tau)\right)d\tau \tag{2.2}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via variational theory. We apply restricted variations to nonlinear term Nu so that we can determine the multiplier. In the homotopy perturbation method, the basic assumption is that the solutions can be written asa power series in p.Now, we apply the homotopy perturbation method,

$$\sum_{n=0}^{\infty} p^{(n)}u_n = u_{0(x)} + p\int_0^t \lambda(\tau)\begin{pmatrix}\sum_{n=0}^{\infty} p^{(n)}L(u_n(\tau)) \\ +N\sum_{n=0}^{\infty} p^{(n)}u_n(\tau)\end{pmatrix}d\tau - \int_0^t \lambda(\tau)g(\tau)\,d\tau \tag{2.3}$$

A comparison of like powers of p gives solutions of various orders.

This method does not resort to linearization or assumptions of weak nonlinearity, the solutions generated in the form of general solution, and it is more realistic compared to the method of simplifying the physical problems.

III EQUATIONS OF TWO-PHASE FLOW THROUGH A POROUS MEDIUM

Water-flooding of oil reservoirs will be described by the differential equations of the simultaneous flow of two immiscible, incompressible liquids through a porous medium, derived below. We use the macroscopic approximation of the mechanics of a continuous medium situated at the same point (in a volume element).The fraction of the volume element occupied by the pore space is the porosity m (Bedrikovetsky, 1993)[16]. The ratio of the volume that a fluid occupies to the pore volume is called the saturation of that fluid. We will consider the part of the pore space occupied by water as will be known as the water-saturation or, more simply, the saturation. The percolation velocities of the phases are defined as their volume rates of flow per unit surface area. The velocities of water and oil are denoted by V_W and V_O respectively. We shall also use the total flow velocity V

$$V = V_o + V_w \tag{3.1}$$

In 1942 Buckley and Leverett presented what is recognized as the basic equation for describing immiscible displacement in one dimension (Buckley and Leverett, 1942) [14]. To describe the simultaneous flow of oil and water in the reservoir, applying Darcy's law, the absolute permeability k used earlier must be replaced by the effective permeabilities $k_O(\sigma)$ and $k_W(\sigma)$ respectively, therefore, the law may by generalized for each of the phases in the form

$$V_w = \frac{-k_w(\sigma)}{\mu_w}\frac{\partial p_w}{\partial x} \tag{3.2}$$

$$V_o = \frac{-k_o(\sigma)}{\mu_o}\frac{\partial p_o}{\partial x} \tag{3.3}$$

Capillary pressure is defined as,

$$p_c(\sigma) = p_o - p_w \tag{3.4}$$

Subtracting equation (3.2) from equation (3.3),

$$\frac{\partial p_o}{\partial x} - \frac{\partial p_w}{\partial x} = -V_o\frac{\mu_o}{k_o} + V_w\frac{\mu_w}{k_w}$$

From equation (3.1),

$$V_o = V - V_w \tag{3.5}$$

Using equation (3.4) and (3.5) we get following equation,

$$\frac{\partial p_c}{\partial x} = -(V - V_w)\frac{\mu_o}{k_o} + V_w\frac{\mu_w}{k_w} \tag{3.6}$$

$$\frac{\partial p_c}{\partial x} = -V\frac{\mu_o}{k_o} + \left(\frac{\mu_o}{k_o} + \frac{\mu_w}{k_w}\right)V_w \tag{3.7}$$

$$\left(\frac{k_o}{\mu_o}\right)\frac{\frac{\mu_o}{k_o}}{\frac{\mu_w}{k_w}+\frac{\mu_o}{k_o}}\frac{\partial p_c}{\partial x} = -\frac{\frac{\mu_o}{k_o}}{\frac{\mu_w}{k_w}+\frac{\mu_o}{k_o}}V + V_w \tag{3.8}$$

Using equation (3.4) we get following,

$$\left(\frac{k_o}{\mu_o}\right)\left(\frac{1}{1+\frac{\mu_w}{\mu_o}\frac{k_o}{k_w}}\right)\left(\frac{dp_c}{d\sigma}\right)\left(\frac{\partial \sigma}{\partial x}\right) = -\frac{1}{1+\frac{k_o}{k_w}\frac{\mu_w}{\mu_o}}V + V_w \tag{3.9}$$

Equation (3.9) can be rewritten as,

$$V_w = f(\sigma)V - D(\sigma)\frac{\partial \sigma}{\partial x} \tag{3.10}$$

Where,

$$f(\sigma) = \frac{1}{1+\frac{k_o(\sigma)}{k_w(\sigma)}\frac{\mu_w(\sigma)}{\mu_o(\sigma)}} \tag{3.11}$$

$$D(\sigma) = -\frac{k_o(\sigma)f(\sigma)}{\mu_o}\left(\frac{dp_c(\sigma)}{d\sigma}\right) \tag{3.12}$$

For the simplest case of horizontal flow, with negligible capillary pressure [2], the expression in Eq. (3.10) reduces to

$$V_w = f(\sigma)V \tag{3.13}$$

For displacement in a horizontal reservoir, the Buckley-Leverett function is defined by eq. (3.13).And so $f(\sigma)$, is equal to the ratio of the mobility of water to the sum of the mobilities of the two phases

Denoting m as porosity of porous media, the mass balance equation has the following form,

$$\frac{\partial V_w}{\partial x} + m\frac{\partial \sigma}{\partial t} = 0 \tag{3.14}$$

Substituting value of equation we get,

$$Vf'(\sigma)\frac{\partial \sigma}{\partial x} + m\frac{\partial \sigma}{\partial t} = 0 \tag{3.15}$$

This resulting equation which describes the two-phase flow through a porous medium.[2] Equation (3.14) can be expressed as,

$$\frac{\partial \sigma}{\partial t} + \frac{V}{m}f'(\sigma)\frac{\partial \sigma}{\partial x} = 0 \tag{3.16}$$

$$\frac{\partial \sigma}{\partial t} + \frac{V}{m}\frac{\partial f(\sigma)}{\partial x} = 0 \tag{3.17}$$

$$\text{where } f'(\sigma) = \frac{df(\sigma)}{d\sigma}, \quad \frac{\partial f(\sigma)}{\partial x} = \frac{df(\sigma)}{d\sigma}\frac{\partial \sigma}{\partial x} \tag{3.18}$$

Here,

k_w : the effective permiability of water

μ_w : water viscosity

k_o : the effective permiability of oil

μ_o : oil viscosity

k : absolute permeability

k^*_w : relative permeability of water : $k^*_w = \frac{k_w}{k}$

k^*_o : relative permeability of oil: $k^*_o = \frac{k_o}{k}$

The effective permeability of water and oil are dependent on the saturations of each fluid and the sum of the effective pemeabilties is always less than the absolute permeability.
We introduce dimensionless variables of length and time,

$$X = \frac{x}{L}, \quad T = \frac{Vt}{mL} \tag{3.19}$$

Where L, the length of the reservoir and mL is the pore volume for a reservoir of unit cross-sectional area.
The dimensionless time T can be interpreted as the ratio of the volume of liquid injected into the reservoir by time t to the pore volume of the reservoir. From equation (3.19).

$$x = LX, \quad t = \frac{mL}{V}T$$

$$\partial x = L\partial X, \quad \partial t = \frac{mL}{V}\partial T$$

Substituting these in equation (3.17) gives

$$\frac{V}{mL}\frac{\partial \sigma}{\partial T} + \frac{V}{mL}\frac{\partial f(\sigma)}{\partial X} = 0 \tag{3.20}$$

$$\frac{\partial \sigma}{\partial T} + \frac{\partial f(\sigma)}{\partial X} = 0 \tag{3.21}$$

Thus, in the large, oil displacement by water can be described by the equation (3.19) in the unknown $\sigma(X,T)$.

IV SOLUTION BY VARIATIONAL HOMOTOPY PERTURBATION METHOD

$$f(\sigma) = \frac{\sigma^2}{2}$$

Equation (3.21) reduced to

$$\frac{\partial \sigma}{\partial T} + \sigma\frac{\partial \sigma}{\partial X} = 0 \tag{4.1}$$

$$\sigma(X,0) = \sigma_0 = \exp(-X) \tag{4.2}$$

Applying Variational Homotopy Perturbation method to the above problem

We construct the correction functional for equation (4.1) as,

$$\sigma_{k+1}(t) = \sigma_k(t) + \int_0^t \left[\lambda(\tau)\left(L\sigma_k(\tau) + N\tilde{\sigma}_k(\tau) - g(\tau)\right)\right]d\tau \tag{4.3}$$

Now substituting the value of Lagrange's multilpier

$$\sigma_{k+1}(t) = \sigma_k(t) - \int_0^t \left[\frac{\partial \sigma_k}{\partial T} + \sigma_k\frac{\partial \sigma_k}{\partial X}\right]dT \tag{4.4}$$

$$\left(\sigma_0 + p\sigma_1 + p^2\sigma_2 + ...\right) = \sigma_0(t) - \int_0^t \begin{bmatrix} \frac{\partial}{\partial T}\left(\sigma_0 + p\sigma_1 + p^2\sigma_2 + ...\right) + \\ \left(\sigma_0 + p\sigma_1 + p^2\sigma_2 + ...\right) \\ \frac{\partial}{\partial X}\left(\sigma_0 + p\sigma_1 + p^2\sigma_2 + ...\right) \end{bmatrix} dT$$

Comparing the coefficient of like powers of p, we have

$$p^0 : \sigma_0(x,t) = e^{-x}$$

$$p^1 : \sigma_1(x,t) = \int_0^t \sigma_0 \frac{\partial}{\partial x}\sigma_0 d\tau = e^{-2x}T$$

$$p^2 : \sigma_2(x,t) = \frac{3}{2}e^{-3X}T^2 + \frac{2}{3}e^{-4X}T^3$$

$$p^3 : \sigma_3(x,t) = 2e^{-4X}T^3 + \frac{65}{24}e^{-5X}T^4 + \frac{43}{20}e^{-6X}T^5 + \frac{7}{6}e^{-7X}T^6 + \frac{16}{63}e^{-8X}T^7$$

So for the numerical purpose we used few approximations and get the following values of σ(X,T) for different values of X and T.

.

V CONCLUSION

A quasi linear first order partial differential equation for two phase flow through a porous medium has been solved using modified variational homotopy perturbation method. It has been observed that the saturation of water is decreasing for various values of time, this resembles with the physical phenomena of the problem. Notwithstanding the same can be concluded by considering other types of function of saturation. It can be easily concluded that the method is elegant and reduces computational work with high accuracy.

TABLE I Values of saturation for different values of time and distance

X\T	0.01	0.02	0.03	0.04	0.05
0.1	0.9131	0.9217	0.9304	0.9394	0.9486
0.2	0.8255	0.8325	0.8396	0.8469	0.8543
0.3	0.7464	0.7520	0.7578	0.7638	0.7698
0.4	0.6749	0.6795	0.6842	0.6890	0.6939
0.5	0.6102	0.6140	0.6179	0.6218	0.6258

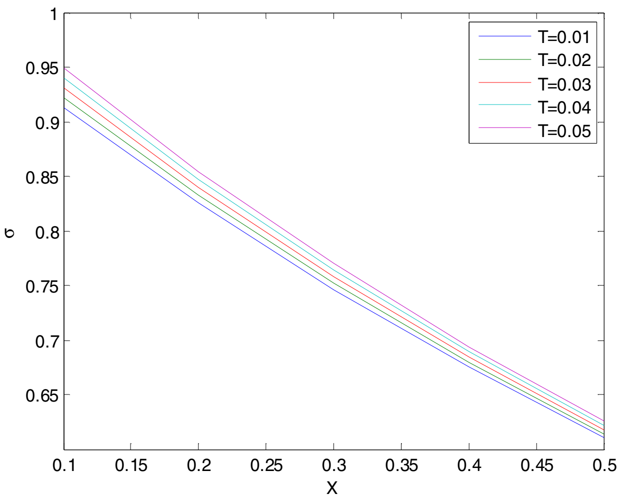

Figure 1.Saturation of water σ(X,T) at different times .

REFERENCES

[1] M.A. Abdou, A.A. Soliman, " Variational iteration method for solving Burger's and coupled Burger's equations", *J. Comput. Appl. Math*. vol 181 (2).pp.245-251,2005.

[2] M. Al-Manna and N.S. Khabeevi, "Mathematical Modeling of Oil Recovery", *Int. J. of Pure and Applied Mathematics*, vol77(2),pp.191-198. 2012

[3] N. Pamuk, " Series solution for porous medium equation with a source term by Adomain decomposition method". *App. Math and Comput*. vol 178,pp.480-485, 2006.

[4] R. Fucik, T. H. Illangasekare, "Evaluation of Saturation-Dependent Flux On Two-phase Flow Using Generalized Semi-Analytic Solution", *Proceeding of Czech-Japanese in Applied Mathematics*,pp.23-35.

[5] J.H. He, "A New approach to nonlinear partial differential equations", *Commun. Nonlinear Sci. Numer. Simul*. vol 2(4) ,pp.203-205,1997.

[6] J.H. He, "Homotopy perturbation method: A new nonlinear technique",*Appl. Math. Comput.* vol.135, pp.73-79,2003.

[7] J.H. He, "Variational iteration method- a kind of non linear analytic technique: Some examples", *Int. J. Nonlinear Mech.* vol. 34,pp. 699-708,1999.

[8] J.H. He, "Some asymptotic methods for strongly nonlinear equation". *Int. J. Mod. Phys.*, vol.20 (20),pp.1144-1199,2006.

[9] J.H. He., " Variational iteration method- Some recent results and new interpretations". *J. Comput. Appl. Math*.,vol. 207.pp. 3-17,2007.

[10] J.H. He and X. Wu, "Variational iteration method: New developments and applications". *Comput. Math. Appl.*, vol.54,pp. 881-894,2007.

[11] A. Ghorbani, and J.S. Nadjfi, "He's homotopy perturbation method for calculating Adomian's polynomials". *Int. J. Nonlin. Scne. Num. Simul.*,vol.8 (2), pp.229-332,2007.

[12] M.A Noor,. and S.T. Mohyud-Din, "Variational iteration method for solving higherorder nonlinear boundary value problems using He's polynomials". *Int. J. Nonlin. Sci. Num. Simul.*,vol.9 (2),pp.141-157, 2008.

[13] M.A.Noor and S.T. Mohyud-Din, "Modified Variational Iteration Method for Goursat and Laplace Problems", *World Applied Sciences Journal* ,vol.4 (4),pp.487-498, 2008 .

[14] S.E. Buckley, M. Leverett, "Mechanics of fluid displacement in sands", *Trans.AIME*, vol.146 ,1942,pp. 107-116.

[15] G.I. Barenblatt, V.M. Entov, V.M. Ryzhik, *Theory of Fluid Flows Through Natural Rocks*, Kluwer Academic Publisher, London (2009).

[16] P. Bedrikovetsky, *Mathematical Theory of Oil and Gas Recovery*, Kluwer Academic Publisher, London ,1993.

Human Development Index – Measurements, Changes and Evolution

Samir K Mahajan

***Abstract*--The Human Development Index (HDI) was introduced in 1990 by UNDP in its First Human Development Report. The index is used for bringing forward the human development profile of countries of the world and thus in ranking and categorizing them on the basis of HDI scores. Since then the HDI has attracted great deal of attraction from critics, policy makers, development thinkers and its advocates. In reaction to the diversified views and opinion, the methodology of the HDI has undergone certain changes from time to time. This paper is a humble effort to throw light on those changes.**

Key words: Development, capabilities, wellbeing, Human Development, HDI

I. INTRODUCTION

The Human Development Index (HDI) is an attempt to reflect human well-being and development experiences of people as individuals, and members of a community, state or nation beyond per capita income or any of its variants. It is the natural progression of a new development discourse called 'human development paradigm' introduced by UNDP in 1990 in its first Human Development Report. Drawing on Sen's ideas of human capabilities and functionings, the Report reject income as the sole yardstick of development, and put people and its well-being at the centre of development strategy. The Human Development Report (HDR)1990 defined human development as a "process of enlarging people's choices' by expanding human capabilities and functionings as in [7]. The most critical of these wide-ranging choices are to live a long and healthy life, to be educated and to have access to resources needed for a decent standard of living.. If these essential choices are not available, many other opportunities remain inaccessible. Rejecting income as the sole measure of well-being/ development, HDR 1991 views that income is one aspect of these choices – and an extremely important one but it is not the sum-total of human life and we do not need unlimited income to lead a good life as [2], [3] and [8].

Thus, along with notion of human development, the UNDP also introduced the HDI by incorporating there basic dimensions of life such as: knowledge, longevity and income as in [7]. The index is used for bringing forward the human development profile of countries of the world and thus in ranking and categorizing them on the basis of HDI scores.

Since then, the HDI has attracted great deal of attraction from critics, policy makers, development thinkers, activists as well as its advocates. In reaction to the diversified views and opinions, the methodology of the HDI has undergone several changes, from time to time, with respect to indicators, formulas, goalposts and others. The purpose of this is to underline different changes that have been introduced HDI, and the evolution that has taken place in it.

II. DIMENSIONS AND INDICATORS

Since its inception, the HDR has been incorporating three basic dimensions of human life in measuring human development. The dimensions that the UNDP has identified in the construction of HDI are longevity, knowledge, and command over resources. However, though, the basic human dimension that has been considered has remained the same, indicators to measure them has changed from time to time.

- *Longevity – the choice/ability to lead a long and healthy life*

Longevity component reflects the choice to lead a long and healthy life. UNDP has used expectation of life at birth to measure longevity. The importance of life expectancy lies in the common belief that a long life is valuable in it and is a function of many variables such as adequate nutrition and good health which are closely associated with higher life expectancy. With expansion in health facilities and increase in availability of food, life expectancy is likely to go up. This association makes life expectancy an important indicator of human development, especially in view of the present lack of comprehensive information about people's health and nutritional status as in [7].

- *Knowledge – the choice/ability to acquire knowledge and be educated*

Knowledge dimension mirrors the choice to acquire knowledge and be educated. Literacy is regarded as the first step in ones' accessibility to knowledge and education as in [7]. Initially in 1990, only adult literacy at 15 years and above was considered to indicate knowledge dimension. Later on, in

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1991, knowledge component was broadened by combining mean years of schooling (one-third weight) with adult literacy rate (two-third weight) in order to give a sense of educational attainment to people not covered by adult literacy rate. However, since 1995, mean years of schooling has been dropped due to lack of updated data and combined gross enrolment ratio[i] has been used instead as in [1]. Knowledge dimension, thus, has been measured by two education variables: adult literacy rate (two-third weight), and combined gross enrolment ratio(one-third weight)

TABLE I
DIMENSIONS AND INDICATORS

DIMENSION	Indicator	Unit of Measurement
Longevity	1990 onwards Life Expectancy at Birth	Years
Education	In 1990 Adult Literacy Rate (15 Years +)	Percent
	During 1991-1994 2/3 Adult Literacy Rate 1/3 Mean Years of Schooling	Percent Years
	During 1995-2009 2/3 Adult Literacy Rate 1/3 Combined Gross Enrolment Ratio	Percent Percent
	2010 onwards ½ Mean Years of Schooling ½ Expected Years of Schooling	Years Years
Command Over Resources	In 1990 Real GDP per Capita (log)	PPP US$
	During 1991-1993 Real GDP per Capita (adjusted by Atinskon formula with threshold value derived from poverty line))	PPP US$
	During 1994-1998 Real GDP per Capita (adjusted by Atinskon formula with throsold value derived from global average)	PPP US$
	During 1999-2009 Real GDP	PPP US$
	2010 onwards Real GNI per Capita	PPP US$

Source: UNDP Human Development Reports and [1].

[i] Combined Gross Ratio at primary, secondary and tertiary level

till 2009 as in [14] . HDR 2010, however, replaced this combination, and instead used the combination of mean years of schooling[ii] (one-second weight) and expected years of schooling[iii] (one-second) to measure the knowledge dimension [14]).

- *Command over resources – the choice/ability to enjoy a decent standard of living and have a socially meaningful life*

Command over resources satisfies the choice to enjoy a decent standard of living and have a socially meaningful life – economic attainment. To indicate command over resources, the real gross domestic product (GDP) per capita at PPP US$ has been used by UNDP till 2009. According to HDR 1990, selection of GDP per capita as the sole variable of command over resource is due to scarcity of data with respect to many other variables such as: access to land, credit, income and other resources needed for a decent living. Real GDP figures has been further improved by adjusting it by purchasing power parity(PPP) for better approximation of the relative power of the people of different countries to buy commodities to gain command over resources as in [7]. Further, logarithm or adjusted (Atkinson formula)[iv] of real GDP per capita has been introduced to reduce the over bearing influence of income indicator of high-income countries on HDI from time to time as in [1], [9]and [10]. It literary implies that people do not need excessive financial resources to ensure a decent living. However, UNDP started using natural log of real gross national income (GNI) per capita at PPP US$ instead of log of real GDP per capita at PPP US$ since 2010 as in [14]. GNI is equal to GDP less primary income payable to non-resident units plus primary income receivable from non-resident units. Use natural logarithm was used in order to enable greater familiarity with economics literature, although this did not make any significant difference to the results as in [1].

III. SCALING OF THE INDICATORS

The indicators/variables selected to measure the dimensions of human life have different units of

[ii] Average number of years of education received by people ages 25and older in their lifetime based on education attainment levels of the population converted into years of schooling based on theoretical durations of each level of education attended.

[iii] Number of years of schooling that a child of school entrance age can expect to receive if prevailing patterns of age-specific enrolment rates were to stay the same throughout the child's life.

[iv] The Atkinson formula used in the period from 1991 to 1998 is:
$W(y) = y^*$ for $0 < y \le y^*$
$= y^* + 2[(y-y^*)1/2]$ for $y^* < y \le 2y^*$
$= y^* + 2(y^*1/2) + 3[(y-2y^*)1/3]$ for $2y^* < y \le 3y^*$
$= y^* + 2(y^*1/2) + 3(y^*1/3) + \ldots + n[(y-(n-1)y^*)1/n]$ for $(n-1)y^* < y \le ny^*$.
Between 1991 and 1993, the y* was set at the poverty line value of $4,829, while it was replaced by the global averages of $5,120 in 1994 and 1995, $5,711 in 1996, $5,835 in 1997 and $5,990 in 1998 as in [1].

measurement. Standard measurement of life expectancy at birth is in years, income is at monetary units say PPP in US$, literacy rate is in percentage. There arise methodological problems while averaging the numbers with different units. HDI being an average of different variables with different units, normalizing of variables are done through a process of scaling in order to avoid error of measurement. The maximum and minimum values of the variables under consideration are selected for each variable and the difference between them constitutes the scale. Then the difference between maximum and actual value (in case of deprivation) or the difference between actual and minimum value (in case of achievements) is calculated. Finally the ratio of the difference and the scale is set which gives a unit free normalized score for each variable. The normalized value of the variable lies between extreme numbers 0 and 1, in case of either deprivation or achievement whatever may be the case as in [5]

IV. WEIGHTS FOR THE DIMENSIONS AND INDICATORS

The dimensional components have been given equal weights, not because of simplicity but because of the reasoning that all dimensions included in HDI are equally important and desirable in their own right for building human capabilities as in [2], [3], and [5]. However, different weights have been assigned to the various indicators of knowledge dimension from time to time.

V. GOALPOSTS

Construction of dimension index requires setting of goalposts – the maximum and minimum thresholds of indicators. The basic criteria for goalpost fixation by UNDP have been:

- {Maximum, Minimum} = {Observed Value, Observed Value} *during 1990-1993*
- {Maximum, Minimum} = {Fixed Value, Fixed Value} *during 1994-2009*
- {Maximum, Minimum} = {Observed Value, Fixed Value} *2010 onwards*

Till 1993, actual observed threshold values (minimums and maximums) in the best- or worst-performing countries for any particular year were used as goalposts in normalizing the variables. Such differential values had serious limitations as it could not be known whether changes in HDI values were due to better performance or because of changes in the goalposts . To avoid this problem and for a meaningful interpretation of HDI in inter-temporal comparisons, the UNDP set 'fixed normative values' for the different indicators since 1994[5]. These minimums and maximums are not the observed values in the best- or worst-performing countries in current period but the most extreme values observed or expected over a long period of around 60 years. The minimums are those values which were observed historically, going back about 30 years from 1990s. The maximums are the limits of what can be envisioned during next 30 years as in [5] and [11]. On that basis, the maximum average life expectancy at birth for the foreseeable future has been fixed at 85 years while the minimum value for the same indicator was fixed at 25 years. The maximum income/real GDP per capita likely to be achieved by rich countries by 2020 is set at PPP US$ 40000 at and the minimum value is fixed at PPP US$ 200 at 1990 prices. The minimum threshold income was reduced to PPP US$ 100 at since 1995. For adult literacy rate as well as combined gross enrolment ratio, maximum and minimum values are 100 percent and 0 percent respectively during the period 1994-2009 as in [12], [14]and [15].

One major change that is found in setting goalpost values in recent time is that the UNDP has deviated from the rule of setting fixed goalposts (especially in case of maximum values) since 2010 as in [14].. The maximum values are now differential observed values, and the minimum values are fixed values for the different indicators considered in HDI. Besides UNDP has reduced the fixed minimum value for life expectancy at birth to 20 years on the basis of long-run historical evidence provided by Maddison and Riley as in [14].

VI. CONSTRUCTION OF HUMAN DEVELOPMENT INDEX

Formulation of HDI metric is the final stage in HDI construction. By incorporating three basic components of wellbeing such as: health, knowledge and command over resources, UNDP formulates the composite index which is the HDI. HDI can be formulated either in terms of region's deprivation / shortfall or in terms of region's attainment in each of the various selected indicators.

DEPRIVATION METHOD:

Initially ,during 1990-91, HDI was formulated from deprivation perspective. The composite of average deprivation was identified by going through deprivation in each variable; the HDI was then presented as one minus the composite average deprivation [5]. Construction of HDI from deprivation/ shortfall perspective [6] follows the following sequence:

We have, $$\mathrm{I}_{ij} = \frac{\max(X_{ig}) - X_{ij}}{\max(X_{ig}) - \min(X_{ig})}$$

Where, X_{ij} is the actual value of the i^{th} indicator for the country j . $\max(X_{ig})$ and $\min(X_{ig})$ are maximum and minimum value that i^{th} indicator can attain respectively.

I_{ij},(i = 1, 2, 3) is the deprivation index of i^{th} indicator for country j which lies between 0 and 1.

$$I_{ijk} = \sum_{i}^{n} \theta_i I_{ij} \quad \text{Such that} \quad \sum_{i=1}^{n} \theta_i = 1$$

Where; θ_i is the weights for the Dimension Index of the i^{th} indicator and I_{ijk} is the sub- index of the k^{th} dimension(deprivation) ($\forall$ k = 1, 2, 3,…..m)[v]. containing n indicator(s) for the j^{th} country.

By constructing each deprivation index for country j, an average deprivation index I_j for county j across the variables was defined as a simple un-weighted average of the I_{ij}.

$$I_j = \frac{1}{3}\sum_{k=1}^{3} I_{ijk} \qquad \text{Such that} \qquad \sum_{i=1}^{n} \theta_i = 1$$

The shortfall in the HDI for county j was then defined to this average deprivation. The HDI was then presented as one minus the composite average deprivation. Thus, if H_j is the HDI for country j, we have, by definition,

$$1 - H_j = I_j$$

$$\text{Or,} \quad H_j = 1 - I_j$$

ATTAINMENT METHOD:

Over the years with better understanding of issues, academic reactions, policy responses and demand from development activists, some refinements have taken place in HDI in which dimension index has been presented directly in terms of attainment/achievement[5]. Reference [6] shows that the shortfall perspective has some merit to study how far a country has to travel in order to achieve a desirable goal or target. However, it is preferable to express the HDI in terms of attainments rather than shortfall as such formulation is more natural to assess changes in HDI over time. The attainment perspective is more relevant in examining how well a country is doing, where as shortfall perspective is more relevant in looking at the task ahead .

With the required change in methodology, HDI thus has been presented from attainment perspective since 1992 as in [6] and [9].

The dimension index (from attainment perspective) of each indicator can be written as:

$$D_{ij} = \frac{X_{ij} - \min(X_{ig})}{\max(X_{ig}) - \min(X_{ig})}.$$

Where, X_{ij} is the actual value of the i^{th} indicator Xi for the country j. max (X_{ig}) and min(X_{ig}) ,respectively, are highest and lowest goalpost values that i^{th} indicator can be allotted. D_{ij} is the attainment index of the i^{th} indicator ($\forall$ i=1, 2, 3, ……., n)[vi] that lies between 0 and 1 for country j.

Aggregation formula However, in attainment method, UNDP has experimented with two aggregation formulae to arrive at the HDI score. Though arithmetic mean has been the basis of aggregation formula till 2009, UNDP has stopped using it 2010 onwards and instead started using geometric mean for aggregation as in [14]and[15].

Arithmetic mean has been used to arrive at the HDI score till 2009. It goes as follows:

$$H_{ijk} = \sum_{i}^{n} \theta_i D_{ij} \qquad \text{Such that} \qquad \sum_{i=1}^{n} \theta_i = 1$$

Where, θ_i is the weights allotted to i^{th} indicator and H_{ijk} is the sub- index(attainment) of the k^{th} dimension ($\forall$ k = 1, 2, 3,…..m)[vii] containing n indicator(s) for country j attained through arithmetic method..

Finally, HDI (H_j) for the country j is obtained by taking arithmetic mean (simple average) of these n-dimension indices[viii] as shown below.

$$H_j = \frac{1}{n}\sum_{k=1}^{n} H_{ijk}$$

Geometric mean has been used by UNDP in aggregation formula since 2010. The rational behind use of aggregation formulas is that geometric mean is most affected by lowest values. As such, the countries with low HDI scores can have a improved score in favour of their HDI score if geometric mean is used. Use of geometric mean goes as follows:

$$H^{g}_{ijk} = \frac{\left(\prod_i^n D_{ij}^{\theta_i}\right)^{\frac{1}{\sum_{i=1}^n \theta_i}} - \left(\prod_i^n \min(D_{ig}^{\theta_i})\right)^{\frac{1}{\sum_{i=1}^n \theta_i}}}{\left(\prod_i^n \max(D_{ig}^{\theta_i})\right)^{\frac{1}{\sum_{i=1}^n \theta_i}} - \left(\prod_i^n \min(D_{ig}^{\theta_i})\right)^{\frac{1}{\sum_{i=1}^n \theta_i}}}$$

$$\text{Such that} \quad \sum_{i=1}^{n} \theta_i = 1$$

Where; θ_i is the weight assigned to i^{th} indicator. H^{g}_{ijk} is the sub- index of the k^{th} dimension ($\forall$ k = 1, 2, 3,…..m)[ix] containing n indicator(s) for the country j obtained through geometric method.

Finally, HDI (H_j) for the country j is obtained by taking simple geometric mean of the three dimension indices as shown below.

$$H_j = \left(\prod_i^n H^{g}_{ijk}\right)^{\frac{1}{n}}$$

VII. CATEGORIZATION IN LEVEL OF ACHIEVEMENT IN HUMAN DEVELOPMENT

Like the dimension indices, the HDI is also a unit free value which ranges between 0 and 1, and provides a normalized measure of human development. Higher HDI values indicate higher level of achievement of human

[v] There are three dimensions of HDI such as knowledge, good health, and command over resources)

[vi] Such as: life expectancy for longevity, adult literacy rate and combined gross enrolment for education, real GDP per capita PPP US$ for command over resources

[vii] There are three dimensions of HDI such as knowledge, longevity, and command over resources)

[viii] Same as foot note vi

[ix] Same as foot note vi

development. During 1990-2008, countries were categorized as low level (HDI score < 0.5), medium (0.5 ≥HDI ≥0.8) and high (HDI score > 0.8) level of human development respectively as in [12]. A new classification has been introduced since 2009 to categorize the countries the world into four groups such as: very high (HDI score ≥ 0.90), high (0.899 ≥ HDI score ≥ 0.800), and medium 0.799 ≥ HDI≥ 0.500) and low (HDI score ≤ 0.500) level of human development as in [14].

VIII. Conclusion

The HDI has evolved over the years with changes in goalposts, method of aggregation of the variables and in indicators as well. Life expectancy at birth has remained the sole indicator of health and longevity dimension . Education dimension has been indicated by different set of variables in different HDRs. Use of 'mean years of schooling' and 'expected mean years of schooling' in recent years as indicators of knowledge is of recent development. Numerical measurement of these variables is based on certain complex iterations , and is subject to constraint of data availability. Such calculation may suffer when data are not readily available whether at local, regional, national or global level. Real GDP per capital has remained the basic indicator to me measure command over resources for almost two decades. Use of real GNI precipitate is of recent development. However, human development paradigm lies in the eternal faith that income is not the sum total of life. As such, the various HDRs tried to put less emphasis on the variants of income (whether GDP per capita or GNI per capita). Use of Atkinson formula or that logarithm by UNDP in order to lessen the over- influence per capita income in HDI score is an attempt in this direction. Nevertheless, the question still remains, can income be considered in the construction of HDI? Income or any of its variants are means of development, and human wellbeing is the output of development. HDI attempts to measure human well-being and not inputs of well-being, and there remains the contradiction between income and HDI.

IX. References

[1] Jeni Klugman, Francisco Rodríguez and Hyung-Jin Choi, "The HDI 2010: New Controversies, Old Critiques". Human Development Research Paper 2011, UNDP, pp.4-5, 17-20

[2] Mahbub ul Haq, "The Human Development Paradigm" Sakiko Fukuda-Parr and A. K. Shiv Kumar (eds), Readings in Human Development, Oxford University Press, New Delhi. , 2006, pp.17-34

[3] [Mahbub ul Haq, "The Birth of Human Development Index". Sakiko Fukuda-Parr and A. K. Shiv Kumar (eds), Readings in Human Development, Oxford University Press, New Delhi, 2006, pp.127-137

[4] S. Alkire, "Dimensions of Human Development" .World Development. Vol. 30, No. 2, 2002, pp. 181-205

[5] Selim Jahan, "Evolution of the Human Development Index". Human Development Report Office, Occasional papers12, 1994. Reproduced in Sakiko Fukuda-Parr and A. K. Shiv Kumar (eds), Readings in Human Development, Oxford University Press, New Delhi, pp.154-156, 163-176

[6] Sudhir Anand and Amartya K. Sen, "Human Development Index: Methodology and Measurement". Human Development Report Office, Occasional papers, UNDP, New York, 1994,pp.7-10

[7] Human Development Report 1990, UNDP, New York, pp. 9- 12, 108-109, 123-180

[8] Human Development Report 1991, UNDP, New York, pp.13,88-90

[9] Human Development Report 1992, UNDP, New York , pp.93-95

[10] Human Development Report 1993, UNDP, New York, pp. 100-101

[11] Human Development Report 1994, UNDP, New York ,pp. pp.90-91, 94

[12] Human Development Report 1995, UNDP, New York , pp.134

[13] Human Development Report 2007/2008, UNDP, New York, pp. 299-201, 356

[14] Human Development Report 2009, UNDP, New York ,pp.15, 176-178, 208

[15] Human Development Report 2010, UNDP, New York, pp.216

Homotopy Analysis Method for Thermal Analysis of Wet Fin with All Nonlinearity Effect

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***Abstract*--This paper presents the analytical solution of steady state governing equation for a rectangular fin subjected to simultaneous heat and mass transfer (i.e. wet fin) across the surface as well as at the tip of the fin. The temperature distribution is obtained by using homotopy analysis method for the insulated as well as for the convective fin tip boundary conditions. Earlier studies on wet fin have been restricted to the analysis of nonlinear governing equation with insulated tip condition only. The present study developed a closed-form solution with consideration of nonlinearity effects in both governing equation as well as in the boundary condition.**

***Index Terms*-- analytical solution; heat transfer and mass transfer; homotopy analysis method; rectangular wet fin.**

I. Introduction

Fins are extended surface that are attached to the primary surface of any equipment to enhance the heat exchange process. Fins are widely used in industries [1] and in variety of engineering applications, viz., heat exchangers [2-5], electronic equipments [6, 7], turbine cooling [8], solar-air heaters [9, 10], extended surfaces [11] etc. In these applications, the temperature of the fin surface is below the dew point of the surrounding air. Thus heat transfers occur from the surrounding air to the primary surface through the fin along with the condensation of water vapor across the fin surface. As a result, surface of the fin becomes wet. Due to which the consideration of simultaneous heat and mass transfer across the wet fin surface is an important aspect for accurate estimation of the local temperature [12]. Apart from material thermal conductivity and surface heat transfer coefficient, thermo-geometric and psychrometric conditions are also equally important from the standpoint of energy exchange across the wet fin surface. Thus the relationship between humidity ratio and saturation temperature of the air is important for estimation of the temperature distribution across the wet fin.

Recently, many researchers turn their attention to determine a closed-form analytical solution of the problem involving wet fin by considering linear [13-16] and nonlinear [16, 17] relationship between humidity ratio and saturation temperature of the moist air. However, based on the psychrometric properties of the air, it is more accurate to consider a nonlinear relationship as discussed in [16, 17]. Thus the present study has been carried out to determine the temperature distribution of the wet fin by considering nonlinear relationship between humidity ratio and saturation temperature of the moist air. Earlier analysis on the wet fins [13, 16-18] was only limited to simultaneous heat and mass transfer across the surface of the fin by neglecting the effect of convection at the fin tip. In practical, tip of the fin equally interacts with the surrounding air and possesses the ability to transfer a significant amount of heat due to simultaneous heat and mass transfer events at the tip. This study also considers convection of heat and mass at the tip instead of insulated condition in order to determine the temperature accurately. In connection to the above assumptions, the present article estimates the temperature distribution of the fin involving nonlinear governing equation as well as nonlinear boundary conditions.

From the mathematical viewpoint, the solutions of non-linear equations are quite difficult compared to linear ones especially through analytical approach. Such nonlinear system frequently arises naturally in engineering problems. Due to the presence of nonlinearity in the governing equation as well as in the boundary condition, the present boundary value problem (BVP) becomes more complex and is much more difficult to solve analytically. At present, there exist few well known analytical approaches for solving nonlinear problems, viz., perturbation technique, non-perturbation techniques, homotopy perturbation method [19], homotopy analysis method [20] etc.

The present work is based on the analytical solution of governing nonlinear BVP for wet rectangular fin subjected to nonlinear boundary condition using homotopy analysis method (HAM). HAM is a recently introduced analytical method [20] for solving nonlinear equations, which has undergone rapid development since its inception. It is a powerful method which gives acceptable analytical results with convenient convergence and stability. Unlike perturbation approaches, HAM does not depend on a small parameter. As a result, HAM has the added advantage in solving strongly nonlinear equations. Further, as compared to perturbation approach, HAM includes greater flexibility in the selection of appropriate set of basis functions for the solution. It also allows us to control the convergence rate and the region in a much simpler way [20]. HAM has been successfully applied to many nonlinear differential equations including nonlinear heat transfer problems, such as cooling of lumped system with variable specific heat [21, 22], combined

978-1-4673-1719-1/12/$31.00©2013IEEE

convection-radiation lumped system problems [23], conduction with temperature dependent thermal conductivity [23-26] and various other heat transfer problems on dry fins [27, 28].

To the author's knowledge, no study has been reported on fins with simultaneous heat and mass transfer. Therefore consideration of the problem involving wet fin with simultaneous heat and mass transfer is of a particular interest. In this article, HAM is successfully employed on rectangular wet fin subjected to insulated as well as convective boundary conditions to analyze the temperature distribution along the fin surface.

II. MATHEMATICAL FORMULATION

The present study has been carried out by considering a rectangular fin as shown in Fig. 1. The surface temperature of the fin is kept below the dew point temperature of the air. As a result, the dehumidification process takes place across the surface of the fin. Due to which the fin surface is subjected to simultaneous heat and mass transfer. The one-dimensional steady state governing equation for a rectangular fin subjected to simultaneous heat and mass transfer is given [16] as

$$\frac{d}{dx}\left(kA_c\frac{dT}{dx}\right)+P\left[h\left(T_a-T\right)+h_m h_{fg}\left(\omega_a-\omega\right)\right]=0 \quad (1)$$

where, k is the thermal conductivity of the fin material, A_c is the cross-sectional area of the rectangular fin, P is the perimeter of the rectangular fin $(=wt)$, w is the width of the fin, t is the thickness of the fin, T is the temperature, h is the convective heat transfer coefficient, h_m is the mass transfer coefficient, h_{fg} is the latent heat of condensation of water vapor, ω is the humidity ratio and the subscript a denotes ambient condition.

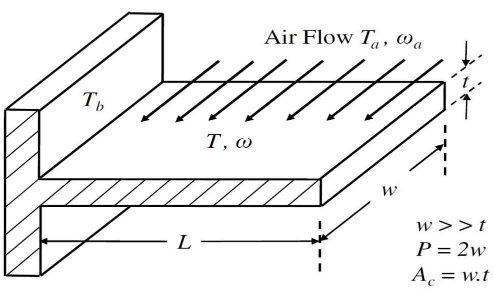

Fig. 1. Schematic of fin geometry.

The Chilton and Colburn [29] relation between the mass transfer coefficient (h_m) and the convective heat transfer coefficient (h) is given as

$$\frac{h}{h_m}=c_p\,\mathrm{Le}^{2/3} \quad (2)$$

where c_p is the humid specific heat of air $\left(=c_{pa}+\omega_a c_{pv}\right)$, c_{pa} is the specific heat of dry air, c_{pv} is the specific heat of water vapor, Le is the Lewis number $\left(=\alpha/D_{mass}\right)$, α and D_{mass} are thermal diffusivity and mass diffusivity of water vapor in air, respectively.

Substituting Eq. (2) in Eq. (1) and rearranging the terms, the governing Eq. (1) can be written as

$$\frac{d^2T}{dx^2}=M^2\left[\left(T-T_a\right)+\xi\left(\omega-\omega_a\right)\right] \quad (3)$$

where M is the fin parameter $\left(=\sqrt{Ph/kA_c}=\sqrt{2h/kt}\right)$, $\xi=\left(h_{fg}/c_p\,\mathrm{Le}^{2/3}\right)$ is the non-dimensional latent heat parameter.

The boundary conditions for the insulated as well as for the convective tip are as follows:

Case I: Insulated tip

$$T(x)\big|_{x=0}=T_b$$
$$\left.\frac{dT(x)}{dx}\right|_{x=L}=0 \quad (4a)$$

Case II: Convective tip

$$T(x)\big|_{x=0}=T_b$$
$$\left.k\frac{dT(x)}{dx}\right|_{x=L}=h\left[\left(T_a-T\right)+\xi\left(\omega_a-\omega\right)\right] \quad (4b)$$

where L is the total length of the fin.

By using non-dimensional parameters $x^*=x/L$ and $\theta=\left(T_a-T\right)/\left(T_a-T_b\right)$, the governing equation (i.e. Eq. (3)) can be written as

$$\frac{d^2\theta}{dx^{*2}}=(ML)^2\left[\theta+\xi\frac{\left(\omega_a-\omega\right)}{\left(T_a-T_b\right)}\right] \quad (5)$$

where ω is the specific humidity of air on the fin surface corresponding to the saturation temperature. In the present analysis the relationship for the specific humidity with the temperature is considered [17] as $\omega=A+BT+CT^2+DT^3$ Where the constants A, B, C and D are obtained by regression analysis using the psychrometric properties of air as illustrated in [17]. Substituting the relationship for ω in Eq. (5) as well as in the boundary conditions (4a) and (4b), we have the following two boundary value problems (BVPs)

$$\frac{d^2\theta}{dx^{*2}}=(ML)^2\left[K_1+K_2\theta-K_3\theta^2+K_4\theta^3\right] \quad (6)$$

subjected to the following boundary conditions

Case I: Insulated tip

$$\theta\left(x^*\right)\big|_{x^*=0}=1$$
$$\left.\frac{d\theta\left(x^*\right)}{dx^*}\right|_{x^*=1}=0 \quad (7a)$$

Case II: Convective tip

$$\theta\left(x^*\right)\Big|_{x^*=0} = 1$$

$$\left.\frac{d\theta\left(x^*\right)}{dx^*}\right|_{x^*=1} = -(ML)^2\psi\begin{bmatrix}K_1+K_2\theta- \\ K_3\theta^2+K_4\theta^3\end{bmatrix} \quad (7b)$$

where $\psi\left(=t/2L\right)$ is the non-dimensional fin aspect ratio; and K_1, K_2, K_3 and K_4 are constants which are given by $K_1=\xi\left(\omega_a\text{-A-B}T_a\text{-C}T_a^2\text{-D}T_a^{\,3}\right)/(T_a-T_b)$,

$K_2=1+B\xi+2C\xi T_a+3D\xi T_a^{\,2}$, $K_3=\xi\left[(C+3DT_a)(T_a-T_b)\right]$ and $K_4=D\xi(T_a-T_b)^2$. Here, x* is the non-dimensional coordinate (i.e. the distance from the fin base), θ is the non-dimensional local fin temperature. The non-dimensional latent heat parameter ξ is found to be varying with various wet conditions or with the variation in relative humidity φ. The relationship between the relative humidity, the specific humidity and the latent heat of condensation of water vapor is given by Kuehn *et al.* [30]. In the present study, the second order nonlinear differential equation, i.e. Eq. (6) subjected to the insulated tip boundary condition (Eq. 7a) as well as nonlinear convective tip boundary condition (Eq. 7b) have been solved with the help of homotopy analysis method (HAM).

III. Basic Idea of HAM

HAM employs the concept of homotopy, a fundamental concept of Topology. To describe the basic idea of HAM, we consider the following differential equation:

$$\mathcal{N}\left[\theta(z)\right]=0 \quad (8)$$

where $\mathcal{N}$ is the non-linear operator, $\theta(z)$ is an unknown function to be solved, and z is the independent variable. By means of generalizing the traditional homotopy method, Liao [20] has developed a new kind of homotopy so called the zero-order deformation equation

$$(1-q)\,\mathcal{L}\left[\Phi(z;q)-\theta_0(z)\right]=q\hbar H(z)\mathcal{N}\left[\Phi(z;q)\right] \quad (9)$$

where $\Phi(z;q)$ is the mapping function of $\theta(z)$, $\theta_0(z)$ is the initial approximation of $\theta(z)$, $\hbar$ is the nonzero auxiliary parameter, $H(z)$ is the auxiliary function, and $\mathcal{L}$ is the linear auxiliary operator.

It is clear that when $q=0$ then $\Phi(z;0)=\theta_0(z)$ and when $q=1$ then $\Phi(z;1)=\theta(z)$, which indicates that as the embedding parameter q increases from 0 to 1, the solution $\Phi(z;q)$ varies from the initial approximation $\theta_0(z)$ to the solution $\theta(z)$ of the Eq. (8). Using the Taylor's theorem, $\Phi(z;q)$ can be expanded with respect to the embedding parameter and is given as

$$\Phi(z;q)=\Phi(z;0)+\sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m\Phi(z;q)}{\partial q^m}\right|_{q=0}q^m \quad (10)$$

Differentiating the zero-th order deformation equation (Eq. 9) m-times with respect to q at $q=0$ and then dividing by factorial of *m*, we have the so called mth order deformation equation

$$\mathcal{L}\left[\theta_m(z)-\chi_m\theta_{m-1}(z)\right]=\hbar H(z)R_m\left(\vec{\theta}_{m-1}\right) \quad (11)$$

where $\theta_m(z)=\frac{1}{m!}\left.\frac{\partial^m\Phi(z;q)}{\partial q^m}\right|_{q=0}$, $\chi_m=\begin{cases}0 & \text{for } m\le 1,\\ 1 & \text{for } m>1\end{cases}$ and

$R_m\left(\vec{\theta}_{m-1}\right)=\frac{1}{(m-1)!}\left[\left.\frac{\partial^{m-1}\mathcal{N}\left[\Phi(z;q)\right]}{\partial q^{m-1}}\right|_{q=0}\right]$. The auxiliary parameter $\hbar$, the auxiliary function *H*, the initial approximation θ_0, and the auxiliary linear operator $\mathcal{L}$ are chosen in such a way that Eq. (10) converges at $q=1$. Therefore, at $q=1$, Eq. (10) becomes

$$\theta(z)=\theta_0(z)+\sum_{m=1}^{\infty}\theta_m(z) \quad (12)$$

The auxiliary parameter provides us a convenient way to control and adjust the rate and the region of the convergence. More detailed information about HAM and its convergence can be found in [20].

IV. Solution of the Rectangular Wet Fin Using HAM

This section is devoted to determine the analytical solution of the second order nonlinear differential equation (Eq. 6) with insulated tip condition (Eq. 7a) as well as nonlinear convective tip boundary condition (Eq. 7b) using HAM. The analytical solution can be obtained by symbolic computation software such as *MATLAB*, *Mathematica* or *Maple*. The present solution is obtained by performing symbolic computation on *MATLAB*.

Using the rule of solution expression [20] and from Eq. (6), it is straightforward to choose a nonlinear operator as well as a linear operator as follows

$$\mathcal{N}\left[\Phi\left(x^*;q\right)\right]=\frac{\partial^2\Phi}{\partial x^{*2}}-(ML)^2\left(K_1+K_2\Phi-K_3\Phi^2+K_4\Phi^3\right) \quad (13a)$$

$$\mathcal{L}\left[\Phi\left(x^*;q\right)\right]=\frac{\partial^2\Phi}{\partial x^{*2}} \quad (13b)$$

with the property $\mathcal{L}\left[c_0+c_1x^*\right]=0$, where c_0 and c_1 are two arbitrary constants.

The solution of the mth order deformation equation (i.e. Eq. (11)) can be written, after using Eqs. (13a) and (13b), as

$$\theta_m\left(x^*\right)=\chi_m\theta_{m-1}\left(x^*\right)+\hbar\int_0^{x^*}ds\int_0^s R_m\left[\vec{\theta}_{m-1}(w)\right]dw+c_1x^*+c_0 \quad (14)$$

where $R_m\left[\vec{\theta}_{m-1}\left(x^*\right)\right]$ is given by

$$R_m\left[\vec{\theta}_{m-1}\left(x^*\right)\right]=\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}\mathcal{N}\left[\Phi\left(x^*;q\right)\right]}{\partial q^{m-1}}\right|_{q=0}\Bigg]$$

$$=\frac{d^2\theta_{m-1}}{dx^{*2}}-(ML)^2\begin{bmatrix}(1-\chi_m)K_1+K_2\theta_{m-1}\left(x^*\right)\\ -K_3\sum_{j=0}^{m-1}\theta_j(x^*)\theta_{m-1-j}(x^*)\\ +K_4\sum_{j=0}^{m-1}\theta_{m-1-j}\sum_{l=0}^{j}\theta_{j-l}(x^*)\,\theta_l(x^*)\end{bmatrix} \qquad (15)$$

with c_0, c_1 are the two integral constants which are determined by using the following boundary conditions:

Case I: Insulated tip

$$\left.\begin{aligned}\theta_m(0)&=0\\ \theta'_m(1)&=0\end{aligned}\right\}\ \text{for } m\geq 1 \qquad (16a)$$

Case II: Convective tip

$$\left.\begin{aligned}&\theta_m(0)=0\\ &\theta'_m(1)=-(ML)^2\psi\begin{bmatrix}K_2\theta_m(1)-K_3\sum_{j=0}^{m-1}\theta_j(1)\theta_{m-1-j}(1)\\ +K_4\sum_{j=0}^{m-1}\theta_{m-1-j}(1)\sum_{l=0}^{j}\theta_{j-l}(1)\theta_l(1)\end{bmatrix}\end{aligned}\right\} \qquad (16b)$$

1) Solution of Rectangular Wet Fin with Insulated Tip Condition

To solve the second order nonlinear equation (Eq. 6) subjected to the insulated boundary condition (Eq. 7a) using homotopy analysis method, we choose the initial approximation as

$$\theta_0\left(x^*\right)=1 \qquad (17)$$

Now, using Equations (17), (14) and (16a), we obtain the values of R_i (i=1,2,...) and then from Eq. (13) we obtain the values of θ_i (i=1,2,3,...) as follows

$$\begin{aligned}\theta_1(x^*)&=\hbar\int_0^{x^*}ds\int_0^{s}R_1\left[\vec{\theta}_0(w)\right]dw+c_1x^*+c_0\\ &=\frac{\hbar}{2}(ML)^2(K_1+K_2-K_3+K_4)\left(1-x^{*2}\right)\end{aligned} \qquad (18a)$$

$$\theta_2(x^*)=\hbar(ML)^2(K_1+K_2-K_3+K_4)\left(1-x^{*2}\right)+$$

$$\frac{\hbar}{2}(ML)^2\begin{bmatrix}(K_4-K_3)+\hbar(K_1+K_2-K_3+K_4)\\ +\frac{5\hbar}{12}(ML)^2\left(K_2^2+2K_3^2+3K_4^2\right)\\ +\frac{5\hbar}{12}(ML)^2\begin{pmatrix}K_1K_2-2K_1K_3+3K_1K_4\\ -3K_2K_3+4K_2K_4-5K_3K_4\end{pmatrix}\end{bmatrix}+\ldots\ldots \qquad (18b)$$

•
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Then the temperature field (θ), the solution of Eq. (6), for the wet fin subjected to the insulated tip condition (Eq. 7a) is

$$\theta(x^*)=\sum_{i=0}^{\infty}\theta_i(x^*)=\theta_0(x^*)+\theta_1(x^*)+\cdots\cdots$$

where $\theta_0(x^*)$ is given by relation (17), $\theta_1(x^*)$ is given by (18a), $\theta_2(x^*)$ is given by relation (18b) and so on. For the computation purpose, instead of an infinite series a finite series has been considered, say of the order m, as

$$\theta(x^*)=\sum_{i=0}^{m}\theta_i(x^*)=\theta_0(x^*)+\theta_1(x^*)+\cdots\cdots+\theta_m(x^*) \qquad (19)$$

2) Solution of Rectangular Wet Fin with Convective Tip Condition

The second order nonlinear differential equation (Eq. 6) subjected to the nonlinear convective tip condition (Eq. 7b) has been solved using homotopy analysis method in this section. The similar mathematical analysis, described above for the case of insulated tip boundary, is followed to obtain the final solution. For nonlinear convective tip boundary condition, the temperature field is obtained as follows

$$\theta(x^*)=\sum_{i=0}^{m}\theta_i(x^*)=\theta_0(x^*)+\theta_1(x^*)+\cdots\cdots+\theta_m(x^*) \qquad (20)$$

where

$$\begin{aligned}\theta_1(x^*)&=\hbar\int_0^{x^*}ds\int_0^{s}R_1\left[\vec{\theta}_0(w)\right]dw+c_2x^*+c_3\\ &=-\frac{\hbar}{2}(ML)^2(K_1+K_2-K_3+K_4)x^{*2}\\ &\quad+\begin{bmatrix}\frac{\hbar}{2}\begin{pmatrix}1+2(ML)^2(K_2-K_3+K_4)\\ -K_2(ML)^4\psi(K_1+K_2-K_3+K_4)\end{pmatrix}\\ -(ML)^2\psi\frac{\hbar^2}{4}K_3\left((ML)^2(K_2-K_3+K_4)+1\right)^2\\ -(ML)^2\psi\frac{\hbar^3}{8}K_4\left((ML)^2(K_2-K_3+K_4)+1\right)^3\end{bmatrix}x^*\end{aligned} \qquad (21)$$

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In the present study m = 4 is considered in both the cases.

To the best knowledge of the authors, at present no work has been reported on the solution of nonlinear governing equation of the wet fin subjected to nonlinear convective boundary equation. Thus, the present work has been carried out by validating the results for linear governing equation of the wet fin with linear boundary conditions given by Sharqawy and Zubair [16]. The comparisons are shown in Figs. 2-3. Figure 2 presents the case when the fin tip is at insulated condition and Fig. 3 presents the case when the fin tip is subjected to convective condition. It has been found that the present results are in satisfactory agreement with those obtained by Sharqawy and Zubair [16].

In the next section, the role of the relative humidity ratio on temperature distribution is discussed in detail.

V. Results and Discussion

The objective of the present work is to use homotopy analysis method for solving the nonlinear governing equation of wet rectangular fin subjected to insulated tip boundary condition as well as nonlinear convective tip boundary condition. Unlike the previous studies on wet fins, the present study is unique since it considers the nonlinear relationship between the humidity ratio and the saturation temperature of the moist air across the fin surface as well as at the fin tip which leads to a complex BVP. In addition, the effect of relative humidity on temperature distribution is discussed.

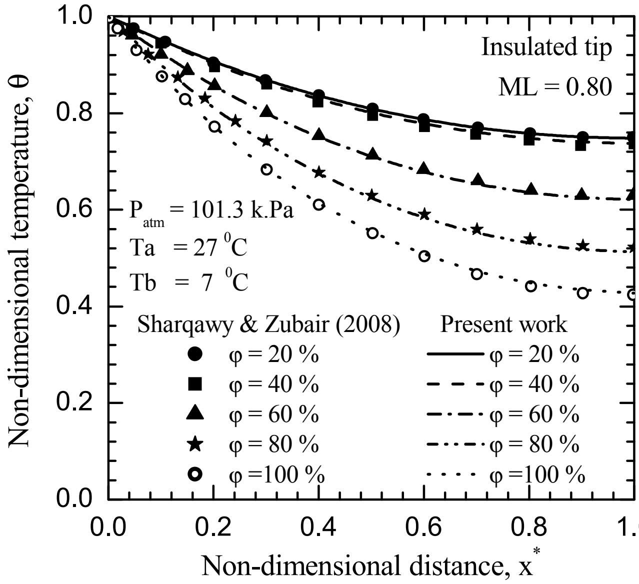

Fig. 2. Comparison of non-dimensional temperature field of wet fin obtained in the present work with that of the results obtained by Sharqawy and Zubair [16] for insulated tip condition.

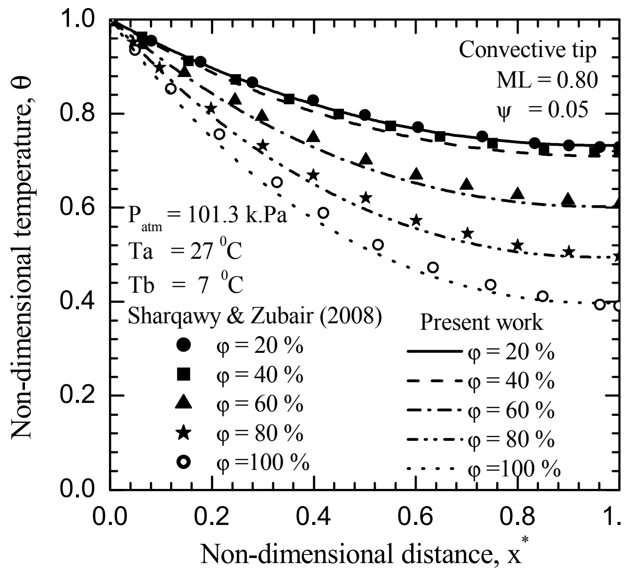

Fig. 3. Comparison of non-dimensional temperature field of wet fin obtained in the present work with that of the results obtained by Sharqawy and Zubair [16] for convective tip condition.

The temperature field of the wet fin for the insulated and for convective tip boundary conditions with due consideration of nonlinear relationship between the humidity ratio and the saturation temperature of the moist air are obtained using HAM and are presented in Figs. 4-5 for different values of relative humidity, viz. $\varphi = 20\%, 60\%, 80\%$ and 100%. The present analysis considers the fin base temperature as $T_b = 7^0$C, the surrounding humid air temperature as $T_a = 27^0$C [31] and air pressure as P_{atm} = 101.3 kPa. Figure 4 depicts the temperature distribution across the fin surface for insulated boundary condition at the tip of the fin. Whereas, Fig. 5 depicts the same for the case of nonlinear convective boundary condition at the tip. It is observed from Figs. 4 and 5 that the consideration of convective boundary condition at the tip leads to significant demarcation in the temperature at

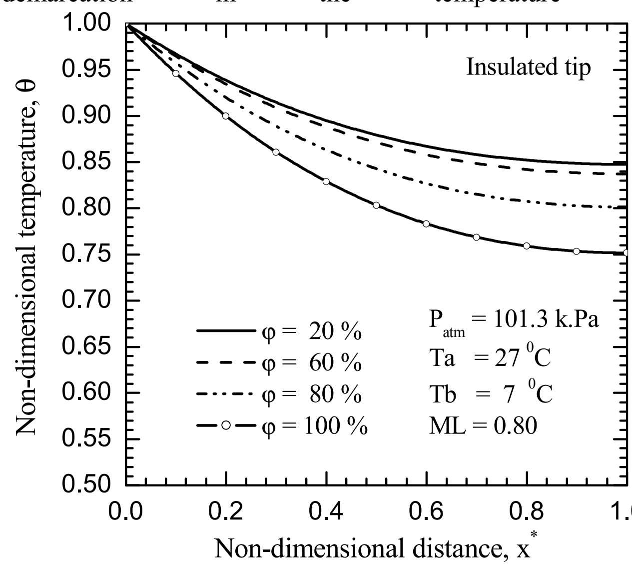

Fig. 4. The non-dimensional temperature field of wet fin for non-linear relationship between the humidity ratio and saturation temperature of air are obtained using HAM for different relative humidity for insulated tip condition.

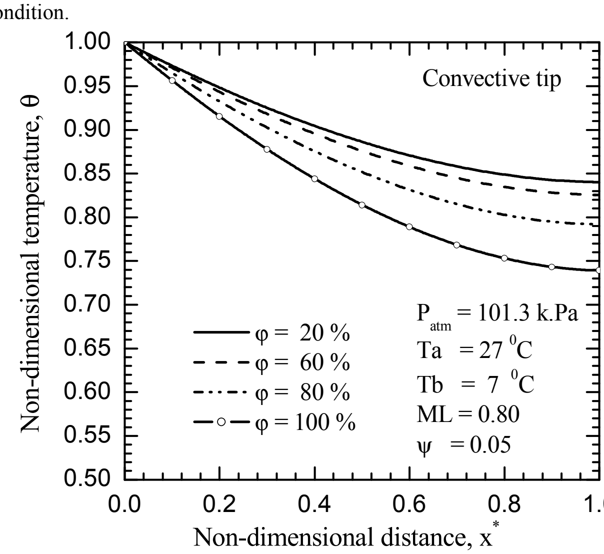

Fig. 5. The non-dimensional temperature field of wet fin for non-linear relationship between the humidity ratio and saturation temperature of air are obtained using HAM for different relative humidity for convective tip condition.

the tip compared to that of insulated boundary condition at the tip (Fig. 4) for different values of relative humidity. Further, with the consideration of convective tip condition the obtained local temperature field across the fin length exhibit a distinctive nature compared to that of the insulated boundary condition at the tip (Fig. 4).

VI. Conclusion

Nonlinear boundary value problems (BVPs) involving heat transfer across a rectangular wet fin subjected to two different boundary conditions, viz., insulated tip and nonlinear convective tip conditions are considered for heat transfer analysis. Solutions (i.e. temperature distributions across the heat transfer surface) of nonlinear BVPs are obtained using homotopy analysis method. For both cases, the temperature distributions across the heat transfer surface are illustrated graphically for various values of relative humidity. Unlike previous studies, the present study suggest that the consideration of insulated tip condition instead of convective tip condition leads to high local temperature at tip, resulting to inappropriate approximation of local temperature field across the fin length. In general, the focus of the present study is to predict the local temperature field accurately across the fin length for the natural heat transfer event where the temperature of the fin tip is also different from the dew point of the adjacent air layer.

VII. References

[1] Q. D. Kern, and A. D. Kraus, *Extended Surface Heat Transfer*, New York: McGraw-Hill, 1972.

[2] K. K. Yau, J. R. Cooper, and J. W. Rose, “Effect of fin spacing on the performance of horizontal integral-fin condenser tubes,” *J. Heat Transfer*, vol. 107, pp. 377-383, 1985.

[3] L. H. Guo, F. Qin, J. P. Chen, and Z. J. Chen, “The Performance Simulation and Optimization on Plate Fin Oil Coolers,” *J. Shanghai Jiaotong Univ.* (Chin. Ed.) , vol. 40, pp. 3-11, 2006.

[4] C. Aprea, and C. Renno, “An air cooled tube-fin evaporator model for an expansion valve control law,” *Math. Comput. Model.*, vol. 30, pp. 135-146, 1999.

[5] S. L. Garrison, B. J. Hardy, M. B. Gorbounov, D. A. Tamburello, C. Corgnale, B. A. VanHassel, D. A. Mosher, and D. L. Anton, “Optimization of internal heat exchangers for hydrogen storage tanks utilizing metal hydrides,” *Int. J. Hydrogen Energy*, vol. 37 pp. 2850-2861, 2012.

[6] K. K. Bodla, J. Y. Murthy, and S. V. Garimella, “Optimization under uncertainty applied to heat sink design,” *J. Heat Transfer*, vol. 135, pp. 1-13, 2013.

[7] Shaeri, M.R., Yaghoubi, M., Jafarpur, K., Heat transfer analysis of lateral perforated fin heat sinks, Appl. Energy, vol. 86, pp. 2019-2029, 2009.

[8] J. Armstrong, and D. Winstanely, “A review of staggered array pin fin heat transfer for turbine cooling applications,” *J. Turbomach.*, vol. 110, pp. 94-103, 1988.

[9] C. D. Ho, H. Chang, R. C. Wang, and C. S. Lin, “Performance improvement of a double-pass solar air heater with fins and baffles under recycling operation,” *Appl. Energy*, vol. 100, pp. 155-163, 2012.

[10] A. P. Omojaro, and L. B. Y. Aldabbagh, “Experimental performance of single and double pass solar air heater with fins and steel wire mesh as absorber,” *Appl. Energy*, vol. 87, pp. 3759-3765, 2010.

[11] A. Bhowmik, R. K. Singla, P. K. Roy, D. K. Prasad, R. Das, and R. Repaka, “Predicting geometry of rectangular and hyperbolic fin profiles with temperature-dependent thermal properties using decomposition and evolutionary methods,” Energ. Convers. Manage., vol. 74, pp. 535-547, 2013.

[12] C. C. Wang,Y. J. Du, Y. J. Chang, W. H. Tao, “Airside performance of herringbone fin-and-tube heat exchangers in wet conditions,” *Can. J. Chem. Eng.,* vol. 77, no.6, pp. 1225-1230, 1999.

[13] M. H. Sharqawy, and S. M. Zubair, "Performance and optimum geometry of spines with simultaneous heat and mass transfer," *Int. J. Therm. Sci.*, vol. 48, no.11, pp. 2130-2138, 2009.

[14] G. Wu, and T. Y. Bong, "Overall efficiency of a straight fin with combined heat and mass transfer," *ASHRAE Transactions*, vol. 100, pp. 367-374, 1994.

[15] M. M. Sala El-Din, "Performance analysis of a partially wet fin assembly," *Appl. Therm. Eng.*, vol. 18, pp. 337-349, 1998.

[16] M. H. Sharqawy, and S. M. Zubair, "Efficiency and optimization of straight rectangular fin with combined heat and mass transfer," *Heat Transfer Eng.*, vol. 29, pp. 1018-1026, 2008.

[17] S. Y. Liang, M. Liu, T. N. Wong, and G. K. Nathan, "Analytical study of evaporator coil in humid environment," *Appl. Therm. Eng.*, vol. 19, pp. 1129-1145, 1999.

[18] B. Kundu, "Approximate analytic solution for performances of wet fins with a polynomial relationship between humidity ratio and temperature," *Int. J. Therm. Sci.*, vol. 48, pp. 2108-2118, 2009.

[19] M. Jalaala, K. Mehravarana, and A. Phillion, "The Effect of Temperature Dependency of Surface Emissivity in 1-D and 2-D Nonlinear Heat Radiation Equations," *Heat Transfer Eng.*, vol. 33, no. 11, pp. 982-990, 2012.

[20] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Boca Raton: Chapman & Hall, 2003.

[21] S. Abbasbandy, "The application of homotopy analysis method to nonlinear equations arising in heat transfer," *Phys. Lett. A*, vol. 360, pp. 109-113, 2006.

[22] V. Marinca, and N. Herişanu, "Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer," *Int. Commun. Heat Mass*, vol. 35, no.6, pp. 710-715, 2008.

[23] G. Domairry, and N. Nadim, "Assessment of homotopy analysis method and homotopy perturbation method in non-linear heat transfer equation," *Int. Commun. Heat Mass*, vol. 35, pp. 93-102, 2008.

[24] R. J. Moitsheki, T. Hayat, and M. Y. Malik, "Some exact solutions of the fin problem with a power law temperature-dependent thermal conductivity," *Nonlinear Anal-Real*, vol. 11, no. 5, pp. 3287-3294, 2010.

[25] F. Khani, R. Ahmadzadeh, and M. S. Hamedi-Nezhad, "A series solution of the fin problem with a temperature-dependent thermal conductivity," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 14, no. 7 , pp. 3007-3017, 2009.

[26] F. Khani, and A. Aziz, "Thermal analysis of a longitudinal trapezoidal fin with temperature-dependent thermal conductivity and heat transfer coefficient," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 15, no. 3, pp. 590-601, 2010.

[27] A. Aziz, and M. Torabi, "Convective-radiative fins with simultaneous variation of thermal conductivity, heat transfer coefficient, and surface emissivity with temperature," *Heat Tran. Asian Res.*, Vol. 41, no. 2, pp. 99-113, 2012.

[28] K. Hosseini, B. Daneshian, N. Amanifard, R. Ansari, "Homotopy Analysis Method for a Fin with Temperature Dependent Internal Heat Generation and Thermal Conductivity," *Int. J. Nonlinear Sci.*, vol. 14, no. 2, pp. 201-210, 2012.

[29] T. H. Chilton, and A. P. Colburn, "Mass transfer (absorption) coefficients," *Ind. Eng. Chem.* Vol. 26, pp.1183-1187, 1934.

[30] T. H. Kuehn, J. W. Ramsey, and J. L. Threlkeld, *Thermal Environmental Engineering*. 3rd ed. Pearson Prentice Hall, 1998.

[31] R. B. Montgomery, "Viscosity and thermal conductivity of air and diffusivity of water vapor in air," *J. Meterology*, vol. 4, pp.193-196, 1947.

Finite Element Model to Study Effect of Source Geometry on Calcium Distribution in Cylindrical Shaped Neuron Cell

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***Abstract*-- The study of calcium diffusion in neuron cells has gained interest among researchers during the last two decades, due to its wide variety of roles in human body like muscles contraction, secretion, metabolism, signal transduction etc. Most of the theoretical investigations on calcium diffusion in neurons have been carried out for one and two dimensional cases by various research workers and that too by incorporating a point source of influx. In order to have more realistic study a three dimensional model of calcium diffusion in a cylindrical shaped neuron cells has been developed in this paper. Apart from the point source, the line and surface sources of an influx of Ca^{2+} have been incorporated in the model. Appropriate boundary conditions have been framed. The region is discretized using three dimensional circular sectored elements. Variational finite element method has been employed to obtain the solution. The numerical results have been used to study effect of point source, line source and surface source of an influx on Ca^{2+} distribution in neuron cell.**

***Index Terms*-- Variational form, coaxial circular sector elements, reaction diffusion equation, diffusion coefficient, calcium influx.**

I. Introduction

CALCIUM is one of the most crucial second messengers for signaling processes in a neuron cell. The calcium signaling in neurons plays an amazing role in regulatory behaviour of human organs. The Cytosolic concentration of free calcium ions (Ca^{2+}) plays a major role in the control of cellular behaviour [1, 5]. Another important element in Ca^{2+} handling is buffers. Buffers are proteins binding most of the Ca^{2+} in a cell (up to 99%). They are present in the cytosol. Depending on their diffusion features, buffers are considered as mobile or immobile. The rate constant of Ca^{2+} binding and fast dissociation rate covers the wide range from slow and fast buffer. The calcium signaling in neurons is regulated by various biophysical processes like diffusion of calcium, reaction of calcium with buffers, calcium influx and number of sources and sinks.

The study of calcium diffusion in neuron cell has attracted great deal of attention among research workers due to the important role of calcium in various cells and processes in human organs. A number of experimental and theoretical investigations [6, 23] are reported in the literature for understanding the processes, components and parameters involved in calcium signaling in neuron and other cells [9, 10, 17, 26]. The general buffered calcium diffusion equation is given by:

$$\frac{\partial[Ca^{2+}]}{\partial t} = D_{Ca}\nabla^2[Ca^{2+}] - k_m^+[B]_\infty([Ca^{2+}] - [Ca]_\infty) \quad (1)$$

M. Narangi, N. Neher Carried out the results of linearized buffered Ca^{2+} diffusion in microdomains and its implications for calculation of $[Ca^{2+}]$ at mouth the channel [19]. Also attempts have been made by Smith et. al, [10, 11, 13] added new dimensions and applied analytical and numerical methods to find solutions to steady state problems of calcium diffusion in single neuronal cell. Smith, Dai, Miura & Sherman [9], have obtained the calcium concentration profiles for steady state case by using perturbation methods. S. Tewari, Pardasani [26] developed the finite element model to study the cytosolic $[Ca^{2+}]$ concentration for two dimensional unsteady states. S. Tewari, Pardasani [27] studied the system of reaction-diffusion equations of Cytosolic $[Ca^{2+}]$ concentration for excess buffer approximation (EBA) and rapid buffer approximation (RBA). A. Tripathi, N. Adlakha [2] developed the finite volume model to study calcium diffusion in spherical shaped neuron cell. A. Tripathi, N. Adlakha [3, 4] has also obtained the calcium distribution in neuron cell using finite element method for one and two dimensions in polar coordinates.

In above investigations the research workers have considered only the point source of influx for calcium diffusion in neuron cells. No work is reported till date on study of calcium diffusion in neuron cell incorporating line and surface sources of influx. The most of the study carried out by the research workers is for one and two dimensional cases for calcium diffusion in neuron cell. In order to have more realistic study there is a need to develop three dimensional models of calcium diffusion in a neuron cell. In view of above a three dimensional finite element model of calcium diffusion in a neuron cell has been developed, in the present paper for a steady state case. To make the study more realistic, the line and surface sources of influx, apart from the point source have been incorporated in the model.

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Excess buffering approximation (EBA) has been also incorporated in the model. A computer program has been developed in MATLAB 7.11 for the problem and simulated on Core i3 processor with 2.13 GHz processing speed, 64-bit machine with 320 GB memory [14]. The effect of different geometrical shapes of these sources on calcium distribution in neuron cells has been studied with the help of numerical results. The mathematical formulation and its solution is presented in the next section.

II. MATHEMATICAL FORMULATION

Calcium buffering reaction in a neuron cell can be put in the form of following equations [9-11, 13, 17] :

$$[Ca^{2+}]+[B_j] \underset{k^-}{\overset{k^+}{\rightleftharpoons}} [CaB_j] \quad (2)$$

where $[B_j]$ and $[CaB_j]$ are free buffer and bound buffer, and 'j' is an index over buffer species. The Calcium kinetics is governed by a set of reaction-diffusion equations which in three dimensional polar cylindrical coordinates are given by [12, 15]:

$$\frac{\partial[Ca^{2+}]}{\partial t}=D_{Ca}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial[Ca^{2+}]}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial[Ca^{2+}]}{\partial\theta}\right)+\frac{\partial^2[Ca^{2+}]}{\partial z^2}\right) +\sum_j R_j+\delta\sigma(r) \quad (3)$$

$$\frac{\partial[B_j]}{\partial t}=D_{B_j}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial[B_j]}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial[B_j]}{\partial\theta}\right)+\frac{\partial^2[B_j]}{\partial z^2}\right)+R_j \quad (4)$$

$$\frac{\partial[CaB_j]}{\partial t}=D_{CaB_j}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial[CaB_j]}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial[CaB_j]}{\partial\theta}\right)+\frac{\partial^2[CaB_j]}{\partial z^2}\right)-R_j \quad (5)$$

Where $R_j=-k_j^+[B_j][Ca^{2+}]+k_j^-[CaB_j]$ (6)

$D_{Ca}, D_{B_j}, D_{CaB_j}$ are diffusion coefficients of free calcium, free buffer, and Ca^{2+} bound buffer, respectively; k_j^+ and k_j^- are association and dissociation rate constants for buffer j respectively. For stationary immobile buffers or fixed buffers $D_{B_j}=D_{CaB_j}=0$. The first term on right hand side is due to fick's law [12] of diffusion, the second term R_j is known as the reaction diffusion term and the third term is source amplitude due to calcium channel. Let us let us assume that there is a single mobile buffer species, i.e. $[B]_j=[B]$ and the Excess Buffer approximation (EBA) [12] i.e. the buffer concentration is present in excess and buffer is constant in space and time, i.e. $[B]=[CaB]=const$

$$D_{Ca}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial[Ca^{2+}]}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial[Ca^{2+}]}{\partial\theta}\right)+\frac{\partial^2[Ca^{2+}]}{\partial z^2}\right) -k_m^+B_\infty([Ca^{2+}]-[Ca^{2+}]_\infty)+\sigma\delta(r)=0 \quad (7)$$

where $[B]_\infty$ and $[Ca]_\infty$ are the buffer concentration and calcium concentration respectively. We assume a cytosolic neuron of cylindrical shape and it has been divided into 24 and 48 coaxial circular sectoral elements [as shown in fig 1 & fig 2 respectively]. The centre of circle at $(r=0,\theta=0)$ and radius of circle is $r=5\mu m$.

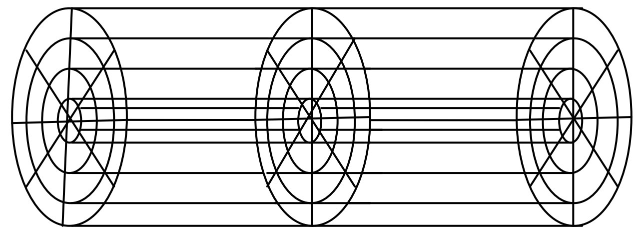

Figure 1 The cylindrical shaped cytosol divided into 48 elements

The influx at boundary is imposed for three cases given below:

Case I: The point source of calcium influx

The point source of calcium influx is assumed to be present at $(r=5,\theta=\pi,z=0)$. Therefore boundary conditions imposed on this point [4, 10, 11, 26] is given by:

$$\lim_{r\to5,\theta\to\pi,z\to0}\left(-2\pi rD_{Ca}\frac{\partial[Ca^{2+}]}{\partial n}\right)=\sigma_{Ca} \quad (8a)$$

where n is normal to the surface

Case II: The line source of calcium influx

The line source of calcium influx is assumed to be present at $(r=5,\frac{3\pi}{4}\le\theta\le\frac{5\pi}{4},z=0)$. The boundary conditions imposed along the curve is given by:

$$\lim_{r\to5,z\to0}\left(-2\pi rD_{Ca}\frac{\partial[Ca^{2+}]}{\partial n}\right)=\sigma_{Ca}$$

$$for(r=5,\frac{3\pi}{4}\le\theta\le\frac{5\pi}{4},z=0) \quad 8(b)$$

where n is normal to the surface

Case III: The surface source of calcium influx

The surface source of calcium influx is assumed to be present at $(r=5,\frac{3\pi}{4}\le\theta\le\frac{5\pi}{4},0\le z\le3)$. The boundary condition imposed on the surface is given by:

$$\lim_{r \to 5,} \left(-2\pi r D_{Ca} \frac{\partial [Ca^{2+}]}{\partial n} \right) = \sigma_{Ca}$$

$$for (r = 5, \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}, 0 \le z \le 3) \qquad 8(c)$$

where n is normal to the surface

For other boundary condition, assumption is the background concentration of $[Ca^{2+}]$ i.e. 0.1μM as it goes far away from the source [2-4, 9, 10, 26, 27].

$$\lim_{r \to 5, \theta \to 0, Z \to 3} [Ca^{2+}] = [Ca]_\infty \qquad (9)$$

The discretized variational form for point source (Case I) of equation (7) & boundary conditions (8(a) & 9) are given by [21-24]:

$$I^{(e)} = \frac{1}{2} \iiint_V \left\{ r \left(\frac{\partial u^{(e)}}{\partial r} \right)^2 + \frac{1}{r} \left(\frac{\partial u^{(e)}}{\partial \theta} \right)^2 + \left(\frac{\partial u^{(e)}}{\partial z} \right)^2 + \frac{1}{\lambda} \left(u^{(e)2} - 2u^{(e)} u_\infty \right) r \right\} dr d\theta dz - \mu^{(e)} \left(\frac{\sigma}{2\pi D_{Ca}} u^{(e)} |_{r=5} \right) \qquad (10)$$

Variational form for line source (Case II) of equation (7) and boundary conditions (8b) and (9) are given by:

$$I^{(e)} = \frac{1}{2} \iiint_V \left\{ r \left(\frac{\partial u^{(e)}}{\partial r} \right)^2 + \frac{1}{r} \left(\frac{\partial u^{(e)}}{\partial \theta} \right)^2 + \left(\frac{\partial u^{(e)}}{\partial z} \right)^2 + \frac{1}{\lambda} \left(u^{(e)2} - 2u^{(e)} u_\infty \right) r \right\} dr d\theta dz - \mu^{(e)} \int_{z_i}^{z_{i'}} \int_{\theta_i}^{\theta_l} \left(\frac{\sigma}{2\pi D_{Ca}} u^{(e)} |_{r=5} \right) d\theta dz \qquad (11)$$

Variational form for surfacesource (Case III) of equation (7) and boundary conditions (8c) and (9) are given by:

$$I^{(e)} = \frac{1}{2} \iiint_V \left\{ r \left(\frac{\partial u^{(e)}}{\partial r} \right)^2 + \frac{1}{r} \left(\frac{\partial u^{(e)}}{\partial \theta} \right)^2 + \left(\frac{\partial u^{(e)}}{\partial z} \right)^2 + \frac{1}{\lambda} \left(u^{(e)2} - 2u^{(e)} u_\infty \right) r \right\} dr d\theta dz - \mu^{(e)} \int_{z_i}^{z_{i'}} \int_{\theta_i}^{\theta_l} \left(\frac{\sigma}{2\pi D_{Ca}} u^{(e)} |_{r=5} \right) d\theta dz \qquad (12)$$

Here, we have used 'u' in lieu of $[Ca^{2+}]$ for our convenience, e=1, 2....48. Also the second term ($\mu_{(e)} = 1$) for e=12, 15 and ($\mu_{(e)} = 0$) for rest of the elements. The following tri linear shape function for the calcium concentration within each element has been taken as [8, 12, 18, 21-25]:

$$u^{(e)} = c_1^{(e)} + c_2^{(e)} r + c_3^{(e)} \theta + c_4^{(e)} z + c_5^{(e)} r\theta + c_6^{(e)} \theta z + c_7^{(e)} rz + c_8^{(e)} r\theta z \qquad (13)$$

$$u^{(e)} = P^T c^{(e)} \qquad (14)$$

Where

$P^T = [1 \quad r \quad \theta \quad z \quad r\theta \quad \theta z \quad rz \quad r\theta z]$ and

$(c^{(e)})^T = [c_1^{(e)} \quad c_2^{(e)} \quad c_3^{(e)} \quad c_4^{(e)} \quad c_5^{(e)} \quad c_6^{(e)} \quad c_7^{(e)} \quad c_8^{(e)}]$

from equation (13) & (14) we get

Where,

$$\overline{u^{(e)}} = \begin{bmatrix} u_i \\ u_j \\ u_k \\ u_l \\ u_{i'} \\ u_{j'} \\ u_{k'} \\ u_{l'} \end{bmatrix} \text{ and } P^{(e)} = \begin{bmatrix} 1 & r_i & \theta_i & z_i & r_i\theta_i & \theta_i z_i & r_i z_i & r_i\theta_i z_i \\ 1 & r_j & \theta_j & z_j & r_j\theta_j & \theta_j z_j & r_j z_j & r_j\theta_j z_j \\ 1 & r_k & \theta_k & z_k & r_k\theta_k & \theta_k z_k & r_k z_k & r_k\theta_k z_k \\ 1 & r_l & \theta_l & z_l & r_l\theta_l & \theta_l z_l & r_l z_l & r_l\theta_l z_l \\ 1 & r_{i'} & \theta_{i'} & z_{i'} & r_{i'}\theta_{i'} & \theta_{i'} z_{i'} & r_{i'} z_{i'} & r_{i'}\theta_{i'} z_{i'} \\ 1 & r_{j'} & \theta_{j'} & z_{j'} & r_{j'}\theta_{j'} & \theta_{j'} z_{j'} & r_{j'} z_{j'} & r_{j'}\theta_{j'} z_{j'} \\ 1 & r_{k'} & \theta_{k'} & z_{k'} & r_{k'}\theta_{k'} & \theta_{k'} z_{k'} & r_{k'} z_{k'} & r_{k'}\theta_{k'} z_{k'} \\ 1 & r_{l'} & \theta_{l'} & z_{l'} & r_{l'}\theta_{l'} & \theta_{l'} z_{l'} & r_{l'} z_{l'} & r_{l'}\theta_{l'} z_{l'} \end{bmatrix}$$

from equation (13) we have

$$c^{(e)} = R^{(e)} \overline{u^{(e)}} \qquad (15)$$

Where $R^{(e)} = P^{(e)-1}$ (16)

Substituting $c^{(e)}$ from equation (16) in equation (14) we get,

$$u^{(e)} = P^T R^{(e)} \overline{u^{(e)}} \qquad (17)$$

Now, the integral $I^{(e)}$ can be in the form

$$I^{(e)} = I_k^{(e)} + I_m^{(e)} - I_s^{(e)} - I_z^{(e)} \qquad (18)$$

Where

$$I_k^{(e)} = \frac{1}{2} \int_{z_i}^{z_{i'}} \int_{\theta_i}^{\theta_k} \int_{r_i}^{r_j} \left[r \left(\frac{\partial u^{(e)}}{\partial r} \right)^2 + \frac{1}{r} \left(\frac{\partial u^{(e)}}{\partial \theta} \right)^2 + \left(\frac{\partial u^{(e)}}{\partial z} \right)^2 \right] dr d\theta dz \qquad (19)$$

$$I_m^{(e)} = \frac{1}{2} \int_{z_i}^{z_{i'}} \int_{\theta_i}^{\theta_k} \int_{r_i}^{r_j} \left[\frac{1}{\lambda} u^{(e)2} r \right] dr d\theta dz \qquad (20)$$

$$I_s^{(e)} = \int_{z_i}^{z_{i'}} \int_{\theta_i}^{\theta_k} \int_{r_i}^{r_j} \left[\frac{1}{\lambda} u^{(e)} u_\infty r \right] dr d\theta dz \qquad (21)$$

$$I_z^{(e)} = \mu^{(e)} \left(\frac{\sigma}{2\pi D_{Ca}} u^{(e)} |_{r=5} \right) \qquad (22)$$

$$\frac{dI^{(e)}}{d\overline{u^{(e)}}} = \frac{dI_k^{(e)}}{d\overline{u^{(e)}}} + \frac{dI_m^{(e)}}{d\overline{u^{(e)}}} - \frac{dI_s^{(e)}}{d\overline{u^{(e)}}} - \frac{dI_z^{(e)}}{d\overline{u^{(e)}}} \qquad (23)$$

$$\frac{dI}{d\overline{u}} = \sum_{e=1}^{N} \overline{M}^{(e)} \frac{dI^{(e)}}{d\overline{u^{(e)}}} \overline{M}^{(e)T} \qquad (24)$$

$$\overline{M}^{(e)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } I = \sum_{e=1}^{48} I^{(e)} \qquad (25)$$

The integral I is extremized with respect to each nodal calcium concentration u_i (i=1, 2.....96). This gives following set of linear algebraic equations [28, 30].

$$\left[\overline{X}\right]_{96\times 96}\left[\overline{u}\right]_{96\times 1}=\left[\overline{Y}\right]_{96\times 1} \qquad (26)$$

The Gaussian Elimination Method has been used to obtain the solution to equation (26).

III. Results and Discussion

The numerical values of physical and physiological parameters used for computation of numerical results are given in Table I:

Table I: List of physiological parameters used for numerical results [9, 10, 26]

Symbol	Parameter	Value
D_{Ca}	Diffusion Coefficient	250$\mu m^2/s$
k^+ (EGTA)	Buffer association rate (Exogenous buffer)	1.5μM^{-1} s^{-1}
$[B_m]_\infty$	Buffer Concentration	50-200 μM
$[Ca^{2+}]_\infty$	Background Ca^{2+} Concentration	0.1 μM
σ	Source amplitude	1-5 pA

1. Point source of influx

Fig. 2 shows the radial and angular calcium distribution in neuron cell for Case I: Point source of influx at z=0 .

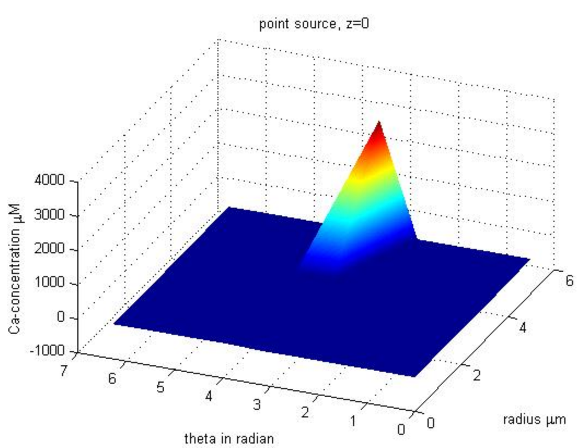

Fig. 2 Calcium distribution in a neuron cell along radial and angular direction for z=0 and Case I: Point source of influx

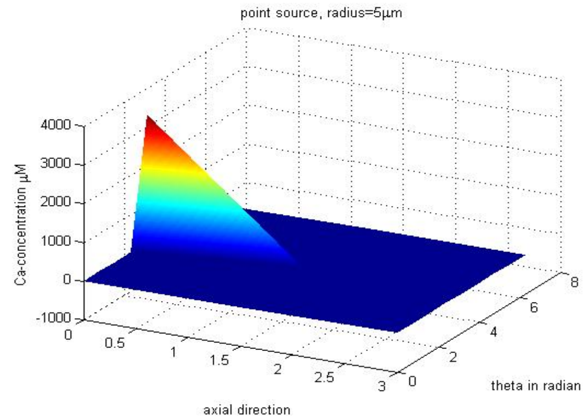

Fig. 3 Calcium distribution in a neuron cell along axial and angular direction for r=5 and Case I: Point source of influx

It can be seen in Fig. 2 that Ca^{2+} concentration is maximum i.e 3300.7 µM at the point source and it decreases as we move away from the source along radial and angular direction and becomes almost constant at background concentration from r=3 µm and θ= 3π/2 onwards.

Fig. 3 shows the axial and angular Ca^{2+} distribution in a neuron cell for Case I: Point source at r=5 µm. The calcium concentration is maximum near the point source at z=0 and θ=π at r=5 and it decreases as we move away along axial and angular direction and becomes almost constant at background concentration of 0.1 µM from z=1.5 µm and θ=π onwards.

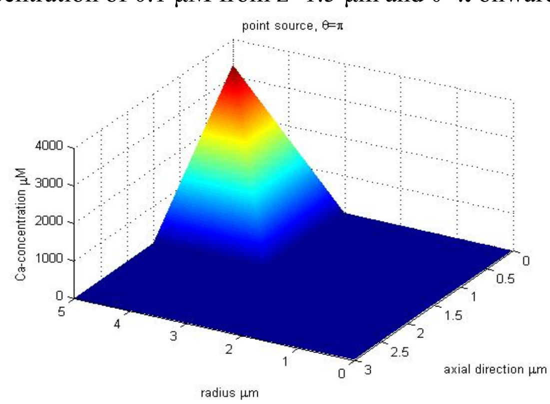

Fig. 4 Calcium distribution in neuron cell along radial and axial direction for **θ=π** and Case I: Point source of influx

Fig. 4 shows the radial and axial Ca^{2+} distribution in a neuron cell at θ=π for Case I. We observe that the calcium concentration is maximum near the point source at r=5µm and θ=π at z=0. When we move away from the point source it decreases and becomes constant from r=3 µm and z=2 µm onwards.

2. Line Source of influx

Fig. 5 shows the radial and angular calcium distribution in neuron cell for Case II: Line source of influx at z=0.

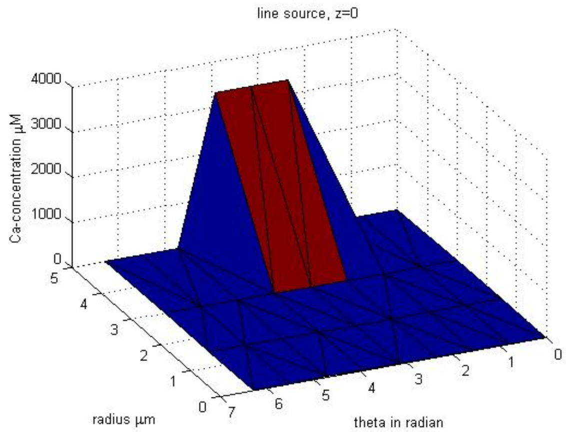

Fig. 5 Calcium distribution in a neuron cell along radial and angular direction for z=0 and Case II: Line source of influx

It can be seen in Figure 5 that Ca^{2+} concentration is maximum i.e. 3300.7 μM at the line source and it decreases as we move away from the source along radial and angular direction and becomes almost constant at background concentration. The effect of line source is seen in Fig. 7 from r=2 μm to r=3 μm and θ= π/2 to θ= 3π/2. Its spread is higher than that in Fig. 2 for Case I.

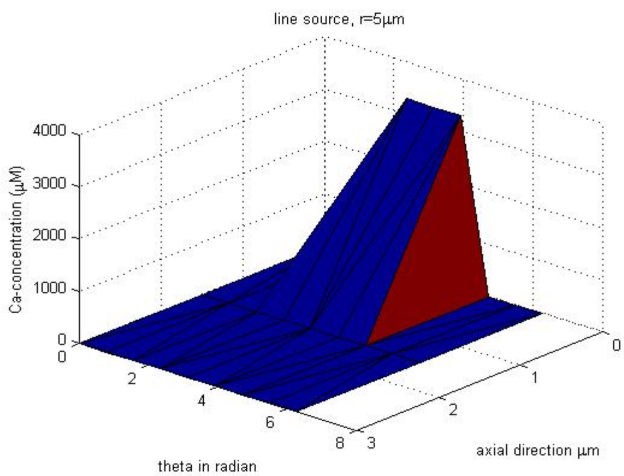

Fig. 6 Calcium distribution in neuron cell along axial and angular direction for r=5 and Case II: Line source of influx

Fig. 6 shows the axial and angular Ca^{2+} distribution in neuron cell for Case II: Line source at r=5 μm. The calcium concentration is maximum near the line source for z=0 and θ=π at r=5 and it decreases as we move away along axial and angular direction and becomes almost constant at background concentration of 0.1 μM. The effect of line source in Figure 8 is seen between θ=π/2 to θ=3π/2 and z= 0 to z=2 μm. When is higher than that in Fig. 3 for Case I: Point source.

3. Surface source of influx

Fig. 7 shows the radial and angular calcium distribution in neuron cell for Case III: Surface source at z=0. The maximum calcium concentration in neuron is 3300.7 μM from θ= 3π/4 to θ= 5π/4. Ca^{2+} concentration decreases as we move away from the source along radial and angular direction and becomes almost constant at background concentration.

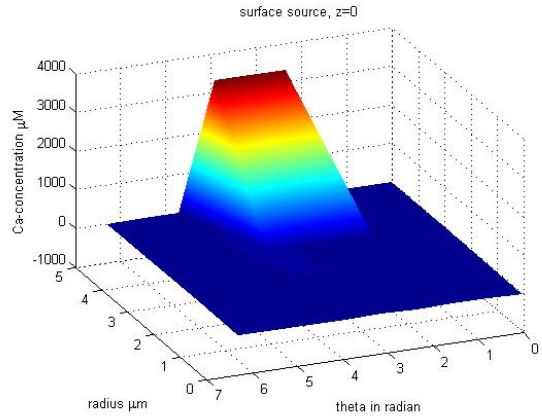

Fig. 7 Calcium distribution in neuron cell along radial and angular direction for z=0 and Case III: Surface source of influx

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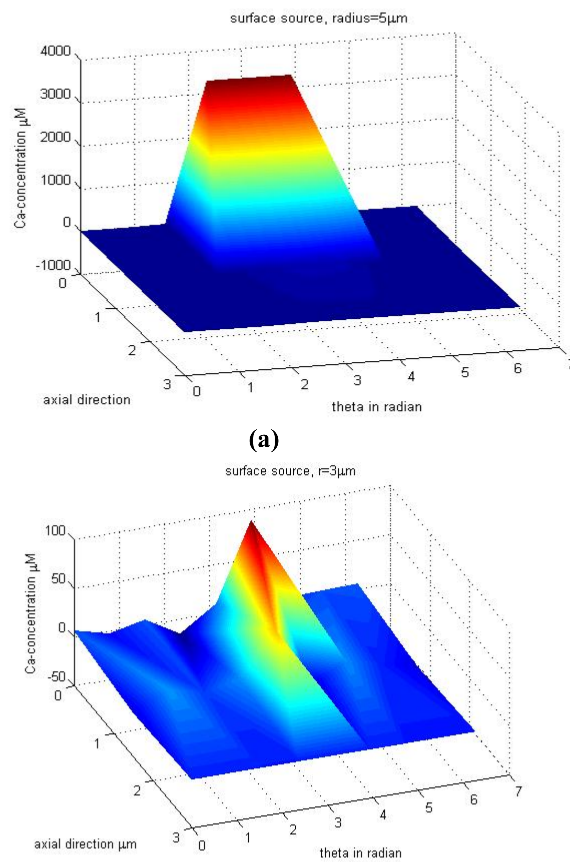

(a)

(b)

Fig. 8 Calcium distribution in a neuron cell along axial and angular direction for r=5 and Case III: Surface source of influx

The effect of surface source is seen in Fig. 9 from r=1 μm to r=4 μm and θ= π/4 to θ= 3π/2. The spread of effect of surface source of influx on Ca^{2+} distribution in fig 8 is more as compared to that in Figure 7 for line source and Figure 1 for point source of influx

Fig. 8 (a) and (b) shows the axial and angular Ca^{2+} distribution in neuron cell for Case III: surface source at r=5 μm and r=3 μm. The calcium concentration is maximum near the surface source for z=0 and θ=π at r=5 & r=3 and it decreases as we move away along axial and angular direction and becomes almost constant at background concentration of 0.1 μM. The effect of surface source in Figure 10 (a) is seen between θ=π/4 to θ=3π/2 and z= 0 to z=2 μm. We observe that significant change in shape of variation of Ca^{2+} distribution in neuron cell due to surface source in comparision to line source and point source of influx.

IV. CONCLUSION

The three dimensional model of Ca^{2+} distribution in neuron cell developed in this paper gives us better picture and makes it possible to incorporate line and surface sources of influx apart from point source. Form the results it can be concluded that the geometry of influx has significant effect on shape of calcium distribution in neuron cell. The three dimensional coaxial circular sectoral elements used in the present finite element model gives us reasonable approximation and reduces the computational work significantly in comparision to that when tetrahedral elements will be used to develop such models. Such models can be developed further to study the relationships among various parameters to get better insights about the components, processes and parameters involved in a neuron cell.

V. REFERENCES

[1] A. David DiGregorio, A. Peskoff, L. Julio, L. M. VergaraFalke, D. Malchow, Measurement of action potential-induced presynaptic calcium domains at a cultured neuromuscular junction understanding of calcium dynamics, Springer- Verlag Berlin Heidelberg NewYork (2003)

[2] A. Tripathi, N. Adlakha Finite volume model to study calcium diffusion in neuron cell under excess buffer approximation International J. of Math. Sci. & Engg. Appls. (IJMSEA). 5 III, 437-447 (2011)

[3] A. Tripathi, N. Adlakha Finite element model to study calcium diffusion in neuron cell in presence of excess buffer for one dimensional steady state Global Journal of Computational Science and Mathematics (GJCSM).; I, 21-30 (2011)

[4] A. Tripathi, N. Adlakha Two dimensional coaxial circular elements in fem to study calcium diffusion in neuron cells Applied Mathematical Sciences 6 no. 10, 455-466 (2012)

[5] CP Fall, Computational cell biology, Springer-Verlag, Berlin Heidelberg NewYork (2002).

[6] D. Holcman, Z. Schuss, Modeling calcium dynamics in dendritic spines SAIM J. of App Maths, 0 1-21 (2005)

[7] E. Neher Concentration profiles of intracellular Ca^{2+} in the presence of diffusible chelator Exp Brain Res 14 80-96 (1986)

[8] F. J. Vermolen, E. Javierre, A finite-element model for healing of cutaneous wounds combining contraction, angiogenesis and closure J. Math. Biol. (2011)

[9] GD. Smith, Miura and Sherman Asymptotic analysis of buffered calcium diffusion near a point source SIAM J. APPL. MATH 61, 5 1816-1838 (2001)

[10] GD. Smith Modeling Intracellular calcium diffusion, dynamics and domains. Modeling in Neuroscience, 339-371 (2004)

[11] GD. Smith Analytical steady-state solution to the rapid buffering approximation near an open Ca^{2+} channel. Biophys J, 71 (6) 3064–3072 (1996)

[12] H. Li Finite element analysis for the axisymmetric Laplace operator on polygonal domains Journal of Computational and Applied Mathematics 235 5155–5176 (2011)

[13] GD. Smith, J Keizer Spark- to-wave transition: salutatory transmission of Ca^{2+} waves in cadiac mayocytes. Biophys Chem, 72 (1-2), 87-100 (1998)

[14] J. Alberty, Kiel, C. Carstensen, Vienna, S. A. Funken, Kiel and R. Klose, Kiel, Matlab implementation of the finite element method in elasticity, Computing 69, 239–263 (2002)

[15] J. Crank The mathematics of diffusion. Oxford, U.K.: Clarendon Press (1975)

[16] J Means, GD Smith shephered shadid fower D Smith, wilson, mazal Wojcikiewicz Reaction diffusion modeling of calcium dynamics with realistic ER geometry Biophys J 91 537-557 (2006)

[17] J. Wanger, J. Keizer Effect of rapid buffers on Ca^{2+} diffusion and Ca^{2+} oscillations Biophys J. 447- 456 (1994)

[18] M. Agrawal, N. Adlakha, K. R. Pardasani Three dimensional finite element model to study heat flow in dermal regions of elliptical and tapered shape human limbs. Applied Mathematics and Computation 217 4129-4140 (2010)

[19] M. Naraghi, E. Neher. Linearized buffered Ca^{2+} diffusion in microdomains and its implications for calculation of $[Ca^{2+}]$ at the mouth of a Ca^{2+} channel. J Neurosci, 17, 6961 (1997)

[20] M. Falke, D. Malchow Understanding of calcium dynamics Springer-Verlag, Berlin Heidelberg NewYork (2003)

[21] M. Gurris, D. Kuzmin and S. Turek Implicit finite element schemes for stationary compressibleparticle-laden gas flows Journal of Computational and Applied Mathematics 235, 5056–5077 (2011)

[22] M. I. A. Othman, I. A. Abbas Fundamental solution of generalized thermo-viscoelasticity using the finite element method Computational Mathematics and Modeling 23, (2012)

[23] N. Volfovsky, H. Parnas, M. Segal, E. Korkotian, Geometry of dendritic spines affects calcium dynamics in hippocampal neurons: theory and experiments Journal of Neurophysiology, 82, 450-462 (1999)

[24] P. Kumar, M. Kumar Finite-element approximation for singularly perturbed nonlinear elliptic boundary-value problems Computational Mathematics and Modeling 23, no. 1, January, (2012)

[25] SS Rao The finite element method in engineering elsevier science and technology books, 92-94 (2004)

[26] S. Tewari, KR. Pardasani Finite element model to study two dimensional unsteady state cytosolic calcium diffusion in presence of excess buffers IAENG International Journal of Applied Mathematics 40:3, IJAM_40_3_01 (2010)

[27] S Tewari, KR Pardasani Finite difference model to study the effects of Na^{+} influx on cytosolic Ca^{2+} diffusion, International journal of Biological and Medical Sciences, 205-209 (2009)

[28] T. Lin, Y. Lin, W.W. Sun, Z. Wang Immersed finite element methods for 4th order differential equations Journal of Computational and Applied Mathematics 235 3953–3964 (2011)

[29] T. Meyer, L. Stryer molecular model for receptor- stimulated calcium 5. spiking, PNAS 85, 14 5051-5055 (1988)

[30] Y. Li, Kaitai Li Pressure projection stabilized finite element method for stokes problem with nonlinear slip boundary conditions Journal of Computational and Applied Mathematics 235 3673–3682 (2011)

Solution of one dimensional ground water recharge phenomenon by Homotopy Analysis Method

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Abstract--**In the present study, theoretical approach used to solve the problem of one dimensional non linear partial differential equation for moisture content of soil. The mathematical formulation leads to a non linear partial differential equation and its solution have been obtained by homotopy analysis method. The analytical results of the equation have been obtained in term of negative exponential terms, which represents the moisture content $\theta(Z,T)$ of soil for any depth $Z,\ T>0$. The results reveal that this method is very efficient and can be applied to other nonlinear problems. All computations have been carried out using some Maple coding and graphical presentation is given.**

***Index Terms*-- Aqueous diffusivity, Homotopy analysis method, Unsaturated porous media.**

I. INTRODUCTION

Since last two decades, the following two major approaches have been considered, (1) All perturbation techniques and (2) Non perturbation techniques. Firstly, all perturbation techniques such as perturbation method [12], Matched asymptotic expansions [29], and Pertubations method by J. Murdock [22] is strongly dependent upon small or large physical parameter. Secondly, non-perturbation techniques such as, the Lyapunov artificial small parameter method [3], δexpansion method employed by A. V. Karmishin et al.[8], and Adomains decomposition method [13], [14], [15] and so on which are formally independent of small or large physical parameters

Based on the homotopy method in topology, Liao proposed such a new kind of analytic technique, known as homotopy analysis method (HAM). The homotopy analysis method has a great advantage that in general its validity it does not depend upon on small or large parameters and it is easy to adjust the convergence region and rate of approximation series. Therefore, the homotopy analysis method handles linear and nonlinear problems without any assumption and restriction. We note here that homotopy perturbation method [20], [21] is a particular case of the HAM [35]. Indeed S. J. Liao [35], [38] makes a compelling case that the Adomain decomposition method, The Lyapunov artificial small parameter method and the δ-expansion method are nothing but special cases of the HAM.

Using one interesting property of homotopy, we can transform any non-linear problem into linear problems. HAM has been applied to nonlinear fluid dynamics problems [36], [37], [38]. The HAM contains the auxiliary parameter $\hbar$, which provides over here a simple way to adjust and control the convergence region of solution series for large values of *t*.

In recent years, this method has been successfully employed to solve many types of linear and nonlinear differential equations in various fields of engineering and science by many authors [34], [25], [19]. Many researchers have been successfully applying homotopy analysis method (HAM) to various nonlinear problems in science and engineering, such as, Tao L Song H [41] obtained analytical approximation of the nonlinear waves in water of finite depth, AS Bataineh et al. [5] has solved time-dependent Emden-Fowler type equations, H. N Hassan and El-Tawil [17] have obtained solution of the two points nonlinear boundary value problems, O Abdulaziz et al.[29] has given a review on time-fractional partial differential equation, H. Hossein Zadeh et al. [16] has given solution of the integral and integrodifferential equations, and A Sami Bataineh et al. [6] had obtained the direct solution of singular higher-order boundary value problems by the homotopy analysis method.

In an environment, most of the water in the ground comes from precipitation that infiltrates downward from the land surface. Groundwater is a key element in many geological and hydro-chemical processes, enhanced oil recovery processes, geotechnical factor conditioning soil and rock behaviour and component of the ecological system which sustains spring discharge, river base flow, lakes and wetlands. The upper layer of the soil is the unsaturated zone, where water is present in varying amounts that change over time, but does not saturate the soil. Below this layer is the saturated zone, where all of the pores, and spaces between soil particles are contains water and empty void of air combination. The top of the surface where groundwater occurs is called the water table. In the fingure1, we can see how the ground below the water table is saturated with water (the saturated zone), which is the upper limit of completely water-saturated soil. In this zone the water saturation varies between 0 and 1 the rest of the pore space normally being occupied by void space. Water flow in this unsaturated zone is complicated by the fact that the soil's permeability to water depends on its water saturation [33]. Aquifers are replenished by the seepage of precipitation that falls on the land, but there are several geologic, meteorologic and human factors that determine the extent and rate to moisture content are filled in unsaturated porous media. Soils have a different porosity and permeability characteristic, which means that moisture, does not spread around the same way in all soil.

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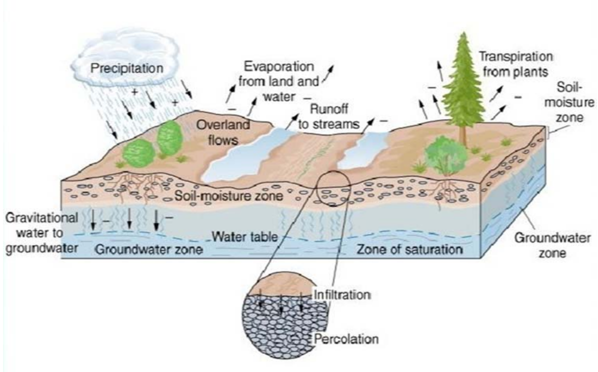

Fig 1: Representation of groundwater recharge phenomenon

The phenomenon of the one dimensional vertical groundwater recharge by spreading is of great important for hydrologist, agriculturists and people associated with water resources sciences. This phenomenon has been discussed by many researchers with different viewpoints. Swartzendruber uses Philip's [32] method to get graphical illustration of the mathematical solution for horizontal water function. A. P. Verma and Mishra [4] have obtained solution by similarity transformation of a unidimensional vertical ground water recharges through porous media. M. Mehta [27] has obtained an approximate solution by the method of singular perturbation technique. He considered the average diffusivity coefficient of the entire range of moisture content and treated as small constant. Hari Prasad et al. [18] had developed a numerical model to simulate water flow through unsaturated zones and study the effect of unsaturated soil parameters on water movement during different processes such as gravity drainage and infiltration. They have worked on a numerical model to simulate moisture flow through unsaturated zones using the finite element method. They examined the sensitivities of different processes such as gravity drainage and infiltration to the variations in the unsaturated soil parameters. This model is also applied to predict moisture contents during a field internal drainage analysis. De Vries and Simmers [11] have discussed processes and challenges principally on recharge of unconfined aquifers, usually the most readily avaliable and affordable source of water in partly arid regions. Faybishenko [10] has given review of the theoretical concepts, has presented the results, and provided perspectives on investigations of flow and transport in unsaturated heterogeneous soils and fracture rock, using the methods of nonlinear dynamics and deterministic chos. T. Patel and M. Mehta [40] studied the ground water recharge by spreading in vertical downward direction. They constitute governing differential equation in form of Burgers equation, with permeability as non linear function of moisture content. M. Mehta and S. Yadav [26] considered an aqueous conductivity directly proportional to the depth, moisture content and inversely proportional to time. They obtained an approximate solution for the problem of vertical groundwater in slightly saturated porous media by using small parameter method. R. Maher and M. Mehta [31] find out solution of one dimensional moisture content problem in porous media by Adomain decomposition method. K. Patel and M. Mehta [24] have given a series solution of the groundwater problem in heterogeneous porous media.

In this paper homotopy analysis method is applied to solve non-linear partial differential equation for the phenomena of the one dimensional unsteady and unsaturated fluid flow through homogeneous porous media. Its solution gives moisture content of the soil at any depth Z for given time $T > 0$. The graph of moisture content versus time T are given at depth Z, which shows at any depth Z moisture content increase as time T increases.

II. STATEMENT OF THE PROBLEM

In the investigated problem, it is considered that groundwater recharge takes place over the large basin of such geological large basin that the sides are limited by rigid boundaries and the bottom by a thick layer of water table. In this case the flow takes place vertically downwards direction up to depth L, (i.e. L= length of large basin), neglecting distribution in other directions (i.e. small amount of water may distribute in other directions but it is very small compared to large size basin) through slightly saturated porous media. Under these circumstances, water from the spreading grounds will flow vertically downward through the unsaturated porous media. It is assumed that the diffusivity coefficient $D(\theta)$ is equivalent to its average value over the whole range of moisture content. We assumed that the permeability of the media varies as square of the moisture content i.e. $k \propto \theta^2$ [28].

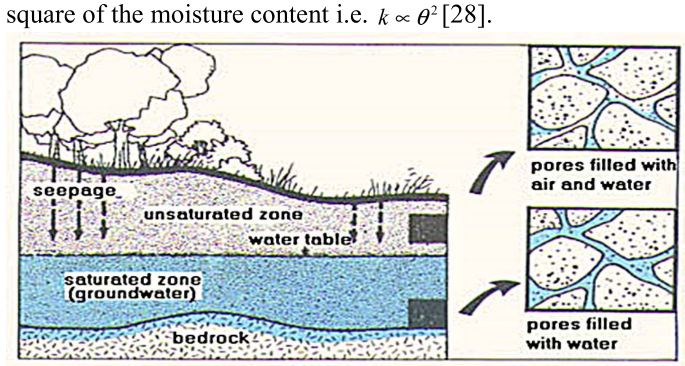

Fig 2: Representation groundwater phenomenon by spreading moisture content in pore space of the soil.

The assumptions have been considered for the present analysis such as the medium is homogeneous and there is no air resistance to the flow (i.e. the porous medium contains only the flowing liquid water and empty voids of air). The air in the void space is approximately at atmospheric pressure i.e. air is stationary. The flowing liquid (water) is considered continuous at a microscopic level, incompressible and isothermal, where the moisture content at the soil surface is considered as time depended function. The saturation of soil or rainfall or irrigation rate is constant and last assumption is Darcy's law and its limitation are also applicable [18].

III. MATHEMATICAL FORMULATION

When any fluid flow vertically downword direction through macroscopically homogenous porous media for small Reynold number the volume of flow of water described by Darcy's law as [23],

$$\vec{V} = -k\nabla H \qquad (1)$$

Where $\vec{V}$ =The volume flux of moisture

k =The coefficient of aqueous conductivity,

∇H =The gradient of the whole moisture potential

Such flow (water) satisfies the equation of continuity as follows,

$$\frac{\partial}{\partial t}(\rho_s \phi S) = -\nabla M \qquad (2)$$

Where ρ_s is the bulk density of the medium on dry weight basis, M is the mass of flux of the water at any time $t \ge 0$.

Considering that water is incompressible and $M = \rho\vec{V}$ and also considering the fact that the water content of soil is given by standard relation with saturation of soil S as $\theta = \phi S$ [9].

Where ϕ porosity and S is saturation of the soil.

Equation (2) reduces to,

$$\frac{\partial}{\partial t}(\rho_s \theta) = -\nabla\left(\rho\vec{V}\right) \qquad (3)$$

Where ρ is the flux density.

Using equation (1) in (3), we get

$$\frac{\partial}{\partial t}(\rho_s \theta) = -\nabla\left(\rho(-k\nabla H)\right) \qquad (4)$$

It is also considered here as that the flow takes place only in vertical downward direction [7], equation (4) reduced to,

$$\rho_s \frac{\partial \theta}{\partial t} = \rho \frac{\partial}{\partial z}\left(k\frac{\partial H}{\partial z}\right) \qquad (5)$$

Further considering the relation between peziometric head H and z, $H = \psi - gz$ [9], where g is gravitational constant, z is vertical direction of flow and ψ is the capillary pressure potential. Hence equation (5) will be,

$$\frac{\partial \theta}{\partial t} = \frac{\rho}{\rho_s}\frac{\partial}{\partial z}\left(k\frac{\partial \psi}{\partial z}\right) - \frac{\rho}{\rho_s} g \frac{\partial k}{\partial z} \qquad (6)$$

The equation (6) can be written as,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) - \frac{\rho g}{\rho_s}\frac{\partial k}{\partial z} \qquad (7)$$

Where, $D(\theta) = \frac{\rho k}{\rho_s}\frac{\partial \psi}{\partial \theta}$, which is called the aqueous diffusivity coefficient. For the sake of simplicity, we assumed that the diffusity coefficient $D(\theta)$ is equivalent to the average value over the whole range of the moisture content of soil as parameter ε i.e. $D(\theta) = \varepsilon$ and the permeability of soil depends on moisture content of soil θ and it may also vary with time and depth. The coefficient of aqueous conductivity was taken as $k = \gamma\theta^2, \gamma = \frac{\rho_s \varepsilon}{2L\rho g}$ [27]. T. Patel and M. Mehta [39] considered the permeability directly proportional to square of moisture content θ.

$$\frac{1}{\varepsilon}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} - \left(\frac{\theta}{L}\right)\frac{\partial \theta}{\partial z} \qquad (8)$$

Now for dimensionless form of equation (8), we choose dimensionless variable,

$$Z = \frac{z}{L} \text{ and } T = \frac{1}{L^2}t \qquad (9)$$

The equation (9) will be reduce to,

$$\frac{\partial \theta}{\partial T} = \varepsilon\left(\frac{\partial^2 \theta}{\partial Z^2} - \theta\frac{\partial \theta}{\partial Z}\right) \qquad (10)$$

The equation (9) is non-linear second order partial differential equation which governs moisture content of soil for the one-dimensional unsteady flow in slightly saturated porous medium in downward direction, over a large basin of length L by geological confirmation and its all sides are limited by rigid boundaries while bottom is confined by a thick larger of water table, where ε is parameter between zero and one.

Mehta [40], Yadav [26] and Meher [31] has obtained solution of the moisture content of soil in terms of negative exponential. Hence for the sake of application of homotopy analysis method, we choose appropriate guess value of solution (11) and boundary condition (12) with respect to this phenomenon as,

$$\theta(Z,T;\varepsilon=0) = \left(1-e^{-Z}\right)T \qquad (11)$$

$$\theta(0,T) = \alpha q(T),\ T > 0, and\ \alpha \ne 0 \qquad (12)$$

Since condition (11) is sufficient to solve problem by homotopy analysis method. Hence we discard the following condition (12) and guess value of solution (11) gives moisture content of soil at top of large basin.

IV. THE SOLUTION WITH HAM

For one dimensional vertical groundwater recharge problem, we have assumed that the moisture content of soil at top of large basin is expressed as

$$\phi(Z,T,\varepsilon) = \left(1-e^{-Z}\right)T + \varepsilon^m \text{, where } \varepsilon = 0 \text{ at top of soil.} \qquad (13)$$

we have applied the homotopy analysis method to the discussed problem. Let us consider the equation (10) as nonlinear partial differential equation

$$N\left[\phi(Z,T;\varepsilon)\right] = 0 \qquad (14)$$

Where N is a nonlinear operator, $\phi(Z,T;\varepsilon)$ is considered as unknown function which represent moisture content θ at any depth Z for given time $T \ge 0$ for $0 \le \varepsilon \le 1$. we use auxiliary linear operator $\Im = \partial/\partial T$ and initial approximation of moisture content of soil is $\theta_0(Z,T) = \left(1-e^{-Z}\right)T$ to construct the corresponding zeroth-order deformation equation. As the auxiliary linear operator which satisfies $\Im\left[C_1\right] = 0$, where C_1 is arbitrary constant. We construct a homotopy as [35],

$$H\left[\phi(Z,T;q);\theta_0(Z,T),H(Z,T),\hbar,\varepsilon\right] = (1-\varepsilon)\left\{\Im\left[\phi(Z,T;\varepsilon)-\theta_0(Z,T)\right] - \varepsilon\hbar H(Z,T)N\left[\phi(Z,T;\varepsilon)\right]\right\} \qquad (15)$$

Enforcing the homotpy (15) to be zero [34],

$$H\left[\phi(Z,T;\varepsilon);\theta_0(Z,T),H(Z,T),\hbar,\varepsilon\right] = 0$$

Establish the zero-order deformation equation of moisture content as [35],

$$(1-\varepsilon)\Im\left[\phi(Z,T;\varepsilon)-\theta_0(Z,T)\right] = \varepsilon\hbar H(Z,T)N\left[\phi(Z,T;\varepsilon)\right] \qquad (16)$$

Where $\theta_0(Z,T)$ denote an initial guess of the exact solution $\theta(Z,T)$ which we want to find, $\hbar \neq 0$ an auxiliary parameter, $H(Z,T) \neq 0$ an auxiliary function, $\varepsilon \in [0,1]$ an embedding parameter and $\Im$ an auxiliary linear operator with the property

$\Im[\phi(Z,T;\varepsilon)] = 0$ when $\theta(Z,T;\varepsilon) = 0$

When $\varepsilon = 0$, the zero-order deformation equation (16) becomes

$$\Im[\phi(Z,T;0) - \theta_0(Z,T)] = 0 \quad (17)$$

Which gives,

$$\phi(Z,T;0) = \theta_0(Z,T) \quad (18)$$

When $\varepsilon = 1$, since $\hbar \neq 0$, $H(Z,T) \neq 0$ the zero-order deformation equation (16) is equivalent to

$$N[\phi(Z,T;1)] = 0 \quad (19)$$

Which is exactly the same as the original equation (14), provided

$$\phi(Z,T;1) = \theta(Z,T) \quad (20)$$

According to (18) and (20) as the embedding parameter ε increases from 0 to 1, solution $\phi(Z,T;\varepsilon)$ varies continuously from the initial guess $\theta_0(Z,T)$ of the moisture content of soil to the solution $\theta(Z,T)$, and its solution is assumed as,

$$\phi(Z,T;\varepsilon) = \phi(Z,T;0) + \sum_{m=1}^{\infty} \theta_m(Z,T)\varepsilon^m \quad (21)$$

Where $\theta_m(Z,T) = \frac{1}{m!} \frac{\partial^m \phi(Z,T;\varepsilon)}{\partial \varepsilon^m}\Big|_{\varepsilon=0}$ (22)

i.e. the moisture content of soil is function of Z, T for any parametric value ε is expressed as, the motiure content of soil at top of large basin $\phi(Z,T;0)$ and sum of moisture contents of soil at different depth layer for different value of parameter ε. If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function $H(Z,T)$ are so properly chosen, the series (21) converges at $\varepsilon = 1$.

Hence the moisture content of soil can be expressed as,

$$\theta(Z,T) = \theta_0(Z,T) + \sum_{m=1}^{\infty} \theta_m(Z,T) \quad (23)$$

and $\theta_m(Z,T)$ can be calculate by equation (28).

This must be one of solution of original non-linear partial differential equation (10) of the motiure content of soil. According to the definition (22), the governing equation can be deduced from the zero-order deformation equation (15). Define the vector

$$\vec{\theta}_n = \{\theta_0(Z,T), \theta_1(Z,T), \ldots \theta_n(Z,T)\}$$

Differentiating equation (16) m times with respect to the embedding parameter ε and then setting $\varepsilon = 0$ and finally dividing them by $m!$, we have the so-called m[th] order deformation equation of the moisture content θ will be as,

$$\Im[\theta_m(Z,T) - \chi_m \theta_{m-1}(Z,T)] = \varepsilon \hbar H(Z,T) R_m(\vec{\theta}_{m-1}, Z, T) \quad (24)$$

Where

$$R_m(\vec{\theta}_{m-1}, Z, T) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(Z,T;\varepsilon)]}{\partial \varepsilon^{m-1}}\Big|_{\varepsilon=0} \quad (25)$$

And, $\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$ (26)

It should be emphasized that $\theta_m(Z,T)$ for $m \ge 1$ is governed by the linear equation (25) with the linear boundary condition that come from original problem, which can solved by symbolic computation software Maple as bellow. The auxiliary function $H(Z,T) = 1$, [35].

According to (20) and taking inverse of equation (24) the equation (24) become,

$$\theta_m(Z,T) = \chi_m \theta_{m-1}(Z,T) + \hbar \Im^{-1}\left[R_m(\vec{\theta}_{m-1}, Z, T)\right] \quad (27)$$

$$R_m(\vec{\theta}_{m-1}, Z, T) = \frac{1}{m!} \frac{\partial^{m-1} N[\phi(Z,T;\varepsilon)]}{\partial \varepsilon^{m-1}} \quad (28)$$

In this way, we get $\theta_m(Z,T)$ for $m = 1,2,3\ldots$ successively by using Maple software as,

$$\theta_1(Z,T) = \frac{1}{6} Th\left(2T^2 e^Z - 2T^2 + 3Te^Z + 6e^{2Z} - 6e^Z\right)e^{-2Z} \quad (29)$$

$$\theta_2(Z,T) = \frac{1}{120} hT \begin{pmatrix} 32hT^4 - 8hT^4 e^{2Z} - 24hT^4 - 25hT^3 e^{2Z} \\ +70hT^3 e^Z + 100hT^2 e^{2Z} - 120hTe^{2Z} - 120he^{2Z} \\ +120he^{3Z} + 40T^2 e^{2Z} - 40T^2 e^Z + 60te^{2Z} - 120e^{2Z} \end{pmatrix} e^{-3Z} \quad (30)$$

……………

Using initial guess value of moisture content from equation (11) and successive moisture content form (27) and (22) etc. and using in equation (23) , we get

$$\theta(Z,T) = 1 - \begin{Bmatrix} \left(e^{-Z}\right)T - \frac{1}{6} hT\left(2T^2 e^Z - 2T^2 + 3Te^Z + 6e^{2Z} - 6e^Z\right)e^{-2Z} \\ -\frac{1}{240} hT \begin{pmatrix} 32hT^4 e^Z - 8hT^4 e^{2Z} - 24hT^4 - 25hT^3 e^{2Z} \\ +70hT^3 e^Z + 100hT^2 e^Z - 120hT^2 e^Z \\ +120hTe^{2Z} + 120he^{3Z} + 120e^{3Z} \\ +40T^2 e^{2Z} - 40T^2 e^z + 60Te^{2Z} - 120e^{2Z} \end{pmatrix} e^{-3Z} + \ldots \end{Bmatrix} \quad (31)$$

where $\theta_1(Z,T), \theta_2(Z,T), \ldots$ are given by equation (29) and (30) respectively represents moisture content of soil at any time T for vertical direction depth Z for $T > 0$. The solution is an infinite series solution, which represents approximate value of moisture content for time $T > 0$. Which is convergent at $\varepsilon = 1$ for auxillary parameter $h = 0.2$ [35].

Thus the moisture content is expressed in terms of exponentials of function of Z and time $T > 0$, which depends on first guess value of solution (11) only.

V. NUMERICAL AND GRAPHICAL SOLUTION

Numerical and graphical presentations of equation (31) have been obtained by using Maple coding. Fig 3 represents the graphs of moisture content $\theta(Z,T)$ vs. depth Z, for T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9,1.0 is fixed, and Table I indicates the numerical values.

TABLE I

MOSISTURE CONTENT $\theta(Z,T)$ FOR DIFFERENT DEPTH Z FOR FIXED TIME T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0.

TIME T	$\theta(Z,T)$ Z=0.1	$\theta(Z,T)$ Z=0.2	$\theta(Z,T)$ Z=0.3	$\theta(Z,T)$ Z=0.4	$\theta(Z,T)$ Z=0.5	$\theta(Z,T)$ Z=0.6	$\theta(Z,T)$ Z=0.7	$\theta(Z,T)$ Z=0.8	$\theta(Z,T)$ Z=0.9	$\theta(Z,T)$ Z=1.0
0.1	0.0152	0.0269	0.0376	0.0472	0.0559	0.0638	0.0709	0.0774	0.0833	0.0885
0.2	0.0344	0.0576	0.0786	0.0975	0.1147	0.1302	0.1443	0.1569	0.1684	0.1788
0.3	0.0576	0.0919	0.1229	0.1511	0.1765	0.1994	0.2201	0.2389	0.2558	0.2711
0.4	0.0849	0.1303	0.1713	0.2082	0.2416	0.2717	0.2988	0.3234	0.3456	0.3656
0.5	0.1167	0.1729	0.2235	0.2691	0.3101	0.3472	0.3806	0.4108	0.4329	0.4625
0.6	0.1531	0.2199	0.2799	0.3339	0.3825	0.4263	0.4657	0.5012	0.5332	0.5621
0.7	0.1942	0.2715	0.3408	0.4029	0.4589	0.5091	0.5543	0.5949	0.6315	0.6644
0.8	0.2404	0.3281	0.4065	0.4767	0.5396	0.5959	0.6467	0.6922	0.7331	0.7699
0.9	0.2919	0.3899	0.4773	0.5553	0.6249	0.6873	0.7431	0.7932	0.8381	0.8785
1.0	0.3492	0.4574	0.5535	0.639	0.7152	0.7832	0.8439	0.8983	0.9469	0.9906

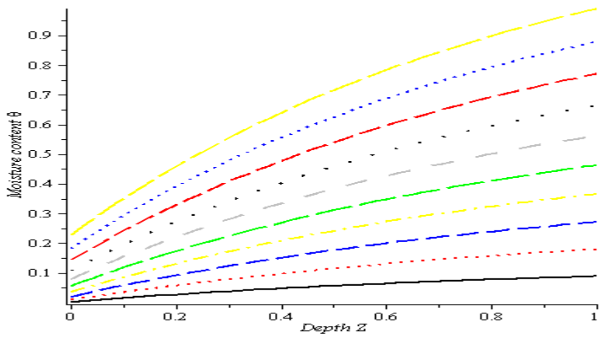

Fig 3: Represents moisture content $\theta(Z,T)$ vs. depth Z for auxiliary parameter $h = 0.2$ and auxiliary function $H(Z,T) = 1$ [35]when T =0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 is fixed.

Table II represents the numerical data. Fig 4 represents the graphs of moisture content $\theta(Z,T)$ vs. time T, Z = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 is fixed.

TABLE II

MOSITURE CONTENT $\theta(Z,T)$ FOR DIFFERENT TIME T FOR DEPTH Z= 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0

TIME T	$\theta(Z,T)$ Z=0.1	$\theta(Z,T)$ Z=0.2	$\theta(Z,T)$ Z=0.3	$\theta(Z,T)$ Z=0.4	$\theta(Z,T)$ Z=0.5	$\theta(Z,T)$ Z=0.6	$\theta(Z,T)$ Z=0.7	$\theta(Z,T)$ Z=0.8	$\theta(Z,T)$ Z=0.9	$\theta(Z,T)$ Z=1.0
0.1	0.0152	0.0269	0.0376	0.0472	0.0559	0.0638	0.0709	0.0774	0.0833	0.0885
0.2	0.0344	0.0576	0.0786	0.0975	0.1147	0.1302	0.1443	0.1569	0.1684	0.1788
0.3	0.0576	0.0919	0.1229	0.1511	0.1765	0.1994	0.2201	0.2389	0.2558	0.2711
0.4	0.0849	0.1303	0.1713	0.2082	0.2416	0.2717	0.2988	0.3234	0.3456	0.3656
0.5	0.1167	0.1729	0.2235	0.2691	0.3101	0.3472	0.3806	0.4108	0.4329	0.4625
0.6	0.1531	0.2199	0.2799	0.3339	0.3825	0.4263	0.4657	0.5012	0.5332	0.5621
0.7	0.1942	0.2715	0.3408	0.4029	0.4589	0.5091	0.5543	0.5949	0.6315	0.6644
0.8	0.2404	0.3281	0.4065	0.4767	0.5396	0.5959	0.6467	0.6922	0.7331	0.7699
0.9	0.2919	0.3899	0.4773	0.5553	0.6249	0.6873	0.7431	0.7932	0.8381	0.8785
1.0	0.3492	0.4574	0.5535	0.639	0.7152	0.7832	0.8439	0.8983	0.9469	0.9906

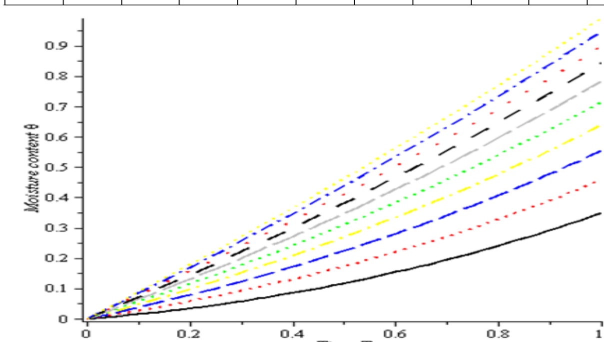

Fig 4: Represents moisture content $\theta(Z,T)$ vs. time T for auxiliary parameter $h = 0.2$ and auxiliary function $H(Z,T) = 1$ [35] when Z =0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 is fixed

VI. CONCLUSION AND DISCUSSION

The equation (31) represents moisture content of soil FOR any depth Z for any time $T > 0$. It is converges for embedded parameter $\varepsilon = 1$ and for auxiliary parameter $h = 0.2$ which is expressed in term of negative exponential terms of Z and time $T > 0$. The moisture content θ will be zero from guess value of the exact solution for $Z = 0$, T=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. The figure number 3 of solution for moisture content θ vs. depth Z shows that moisture content of soil is increasing as depth Z increasing for $T > 0$. From graph it can conclude that for T=0.1 moisture content of soil is linearly increasing as depth Z increasing but when time is increasing and due to different deformation added θ, the moisture content of soil is successively increasing parabolically. Since the equation (11) is parabolic diffusion equation. Hence solution is graphically as well as physically consistent with phenomenon. But the graph of moisture content of soil θ vs. T for given Z=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. The moisture content is also increasing for different time T for given fix value of Z. The moisture content at Z=0.1 is parabolically increasing for different time T but after depth Z is increasing the moisture content of soil is almost linearly increasing with respect to different time T which is shows in graph no 4. From both graph and analytical result (31), we concluded that the moisture content of soil is increasing whne depth as well as time increases. Which is physically fact with phenomenon of ground water recharge in homogenous soil's given in problem.

VI. REFERENCES

[1] A. H. Nayfeh, Perturbation Methods. John Wiley & Sons, New York, 2000.

[2] A. H. Nayfeh, Problems in Perturbation. John Wiley & Sons, New York, 1985.

[3] A. M. Lyapunov, General problem on stability of motion. Taylor & Francis, London, 1992. (English translation).

[4] A. P. Verma and S. K. Mishra, A similarity solution of a one-dimensional vertical groundwater recharge through porous media, Rev. Roum. Scien. Techn. Ser. Mec.Appl. 18 (2), 345-351, 1973.

[5] A. S. Bataineh, MSM Noorani, I. Hashim, Solutions of time-dependent Emden-Fowler type equations by homotopy analysis method. Physics Letters A ;371:72-82, 2007.

[6] A. Sami Bataineh, I. Hashim, et al, Direct solution of singular higher-order BVPs by the homotopy analysis method and its modification, World Applied Sciences Journal 12(6):884-889, 2011.

[7] A. Swaroop and M. N. Mehta, A Solution to the problem of one dimensional flow in unsaturated porous media taking Finite Element Approach, Proceeding of International conference on Mathematical modeling, jan-29-31, 141-143, 2001.

[8] A.V. Karmishin, A. T. Zhukov, and V. G. Kolosov, Methods of dynamics calculation and testing for thin-walled structures. Mashinostroyenie, Moscow, 1990.

[9] B. C. John, Two dimensional unsteady flows in unsaturated porous media, Proc. 1st Int. Conf. Finite Elements in Water Resources, Princeton University, U.S.A., 3.3-3.19, 1976.

978-1-4673-1719-1/12/$31.00©2013IEEE

[10] B. Faybishenko, Nonlinear dynamics in flow through unsaturated fractured porous media, Status and perspectives.Rev. Geophysics, 42, RG2003, doi: 10.1029/ 2003RG00125, 2004.

[11] De varies and Simmers, Ground water recharge: an overview and challenges. Journal of Hydrology, Volume 10(1), 2002.

[12] E. J. Hinch, Perturbation Methods. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1991.

[13] G. Adomian, and G. E. Adomian, A global method for solution of complex systems. Math. Model., 5:521–568, 1984.

[14] G. Adomian, Nonlinear stochastic differential equations. J. Math. Anal. and Applic., 55:441–452, 1976.

[15] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers, Boston and London, 1994.

[16] H. Hossein Zadeh, H. Jafari and S. M Karimi, Homotopy analysis method for solving integral and integro_differential equations, International Journal of Research and Reviews in Applied Sciences2(2).February 2010 .

[17] H. N. Hassan and Magdy A El-Tawil. Solving cubic and coupled nonlinear scho dinger equation using the homotopy analysis method, Int. J. of Appl. Math and Mech. 7 (8): 41-64, 2011.

[18] Hari Prasad, K. S., Mohan Kumar and M. Sekhar, Modeling Flow through unsaturated Zone: Sensitivity to unsaturated Soil Properties, Sadhana, Indian Academy of Sciences, 26 (6),517-528, 2001.

[19] Hayat, T., Khan, M. and Asghar, S., Magnetohydrodynamic flow of an Oldroyd 6–constant fluid. Appl. Math. Comput.Vol. 155, pp. 417–425, 2004.

[20] He, JH: Homotopy perturbation technique. Comput Methods Appl Mech Eng, 178:257–62, 1999.

[21] He, JH: A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int J Non-linear Mech 35:37-43, 2000.

[22] J. A. Murdock, Perturbations: Theory and Methods. John Wiley & Sons, New York, 1991.

[23] J. Bear, Dynamics of fluids in porous media, American Elsevier Publishing Company, Inc, 1972.

[24] K. Patel, M. N. Mehta and T. Patel, A series solution of moisture content in vertical groundwater flow through unsaturated heterogeneous porous media. International eJournal of Mathematics and Engineering 159, 1467-1477, 2012.

[25] M. Ayub, A. Rasheed, and T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method. Int. J. Eng. Sci. Vol. 41, pp. 2091–2103, 2003.

[26] M. N. Mehta and S. Yadav , Classical solution of Non-linear Equations arising in fluid flow through Homogenous Porous Media, 2009.

[27] M. N. Mehta, A singular perturbation solution of one-dimensional flow in unsaturated porous media with small diffusivity coefficient, Proc. FMFP E1 to E4, 1975.

[28] M. S. Joshi, N. B. Desai and M. N. Mehta, One dimenstional and unsaturated fluid flow through porous media, Int. J. of Appl. Math and Mech. 6(18):66-79, 2010.

[29] O. Abdulaziz, I. Hashim and A Saif, Series Solutions of Time-Fractional PDEs by Homotopy Analysis Method, Hindawi Publishing Corporation Differential Equations and Nonlinear Mechanics Volume 2008, Article ID 686512, 16 pages, doi:10.1155/686512, 2008.

[30] P. A. Lagerstrom, Matched Asymptotic Expansions: Ideas and Techniques,volume 76 of Applied Mathematical Sciences. Springer-Verlag, New York, 1988.

[31] R. Meher, M. N. Mehta and S. K. Meher, Adomian decomposition method for moisture content in one dimensional fluid flow through unsaturated porous media, International Journal of Applied Mathematics and mechanics, 6 (7), 13-23, 2010.

[32] R. Philips, Advances in hydro science (edited Ven Te Chow), Acad. Press, New York, 1970.

[33] R.R. Allen, Musick, J.T., Shneider, A.D., Residual deep ploughing effects on irrigation infiltration for Pulman clay loam. Soil Science Society of America Journal 59 (5),1424–1429, 1995.

[34] S. Abbasbandy, Homotopy analysis method for heat radiation equations. Int. Comm. Heat. Mass. Transfer Vol. 34, pp. 380–387, 2007.

[35] S. J. Liao, Beyond perturbation: introduction to homotopy analysis method. CRC Press: Chapman & Hall; 2003.

[36] S. J. Liao, An approximate solution technique which does not depend upon small parameters: a special example. Int. J. Nonlinear. Mech Vol. 30, pp. 371–380, 1995.

[37] S. J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems. Int. J. Nonlinear. Mech. Vol. 34, pp. 759–778, 1999.

[38] S. J. Liao, On the homotopy analysis method for nonlinear problems. Appl. Math. Comput. Vol. 147, pp. 499–513, 2004.

[39] S. J. Liao, Comparison between the homotopy analysis method and the homotopy perturbation method. Appl Math Comput. 169:1186–94, 2005.

[40] T. Patel and M.N. Mehta., The classical solution of Burger's equation arising into the one dimensional groundwater recharge by spreading in porous media, Bulletin of Pure and Applied Sciences Vol.25E-Mathematics & Statistics (No.1) June 2006.

[41] Tao L Song H Chakrabarti S, Nonlinear progressive waves in water of finite depth-ananalytic approximation,Coastal Engineering ;54:825-34, 2007.

Two Dimensional Coaxial Circular Elements in Finite Element Method to Study Calcium Diffusion in Cardiac Myocytes

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Abstract: *The calcium signaling plays an important role in expansion and contraction of Myocytes. This calcium signaling is achieved by diffusion of calcium and buffering mechanisms in cardiac myocytes. In this paper an attempt has been made to develop model calcium signaling in myocytes incorporating diffusion of calcium and excess buffers. The model has been developed for a two dimensional steady state case. Appropriate boundary conditions have been framed. The finite element method has been employed to obtain the solution. The numerical results have been used to study the effect of buffers on calcium distribution in Myocytes.*

Index terms: cardiac myocytes, reaction diffusion equation, excess buffer, finite element method

I INTRODUCTION

The functioning of heart is achieved through expansion and contraction of cardiac myocytes. This expansion and contraction of myocytes is responsible for pumping of blood from heart to arteries. In order to understand the function of heart it is of crucial interest to understand the processes involved in cardiac myocytes. The specific calcium signaling is required to achieve the above function of cardiac myocytes. But this calcium signaling in cardiac myocytes is still not well understood.

Chemical reaction and diffusion are central to quantitative computational biology. As Ca^{2+} ions diffuse away from the mouth of voltage gated plasma membrane through Ca^{2+} channels into the Cytosolic domain of elevated intracellular Ca^{2+} ion activate proteins associated with neurotransmitter release [1] .These Ca^{2+} domain are formed on the presence of ubiquitous Ca^{2+} binding proteins (Troponin-C) of the pre-synaptic terminal. By binding and releasing free Ca^{2+}, endogenous Ca^{2+} binding proteins and other "Ca^{2+} buffers" determine the range of action of Ca^{2+} ions influence the time course of their effect and facilitate clearance of Ca^{2+} [1]. Concentration coupling, Ca^{2+} sparks activated by Ca^{2+} in flux through sarcoleminal Ca^{2+} channels is the "building blocks" of global Ca^{2+} responses that cause contraction. In the present study a mathematical model, for two dimensional steady state calcium diffusion in cardiac myocytes is developed under excess buffer approximation to understand the calcium diffusion in cardiac myocytes. The important parameters like buffers, diffusion coefficients and influx etc has been incorporated in the model.

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II MATHEMATICAL FORMULATION

By assuming a bimolecular association reaction between Ca^{2+} and buffer, we have

$$Ca^{2+} + B \underset{k^-}{\overset{k^+}{\Leftrightarrow}} CaB \tag{1}$$

In equation (1), B represents free buffer, CaB represents Ca^{2+} bound buffer. $k^+$and k^- are association and dissociation rate constants, respectively. If it is further assumed that the reaction of Ca^{2+} with buffer follows mass action kinetics, it can be written the following system of ODEs for the change in concentration of each species is given by [1, 2, 3].

$$\frac{d\left[Ca^{2+}\right]}{dt} = R + J \tag{2}$$

$$\frac{d\left[B\right]}{dt} = R \tag{3}$$

$$\frac{d\left[CaB\right]}{dt} = -R \tag{4}$$

where the common reaction term R, is given by

$$R = -k^+\left[Ca^{2+}\right]\left[B\right] + k^-\left[CaB\right] \tag{5}$$

and J represents Ca^{2+} influx. Both R and J have units of concentration per unit time. Equations (2) to (5) are extended to include multiple buffers and the diffusive movement of free Ca^{2+}, Ca^{2+} bound buffer and Ca^{2+} free buffer. Assuming, Fick's diffusion in a homogeneous, isotropic medium, the system of reaction diffusion equations is written as [1],

$$\frac{\partial\left[Ca^{2+}\right]}{\partial t}=D_{Ca}\nabla^2\left[Ca^{2+}\right]+\sum_i R_i+J \tag{6}$$

$$\frac{\partial\left[B_i\right]}{\partial t}=D_{B_i}\nabla^2\left[B_i\right]+R_i \tag{7}$$

$$\frac{\partial\left[CaB_i\right]}{\partial t}=D_{CaB_i}\nabla^2\left[CaB_i\right]-R_i \tag{8}$$

where the reaction terms, R_i is given by

$$R_i=-k_i^+\left[Ca^{2+}\right]\left[B_i\right]+k_i^-\left[CaB_i\right] \tag{9}$$

where, i is an index over Ca^{2+} buffers. D_{Ca} , D_{B_i} , D_{CaB_i} are diffusion coefficients of free Ca^{2+} , bound calcium and free buffer respectively.

Since Ca^{2+} has a molecular weight that is small in comparison to most Ca^{2+} binding species, the diffusion constant of each mobile buffer is not affected by the binding of Ca^{2+} that is D_{B_i} $=D_{CaB_i}$ $=D_i$.[11, 12] Substituting this in equation (7) & (8) and on summation it gives

$$\begin{aligned}\frac{\partial\left[B_i\right]_T}{\partial t}&=\frac{\partial\left[CaB_i\right]}{\partial t}+\frac{\partial\left[B_i\right]}{\partial t}\\&=D_i\nabla^2\left[CaB_i\right]+D_i\nabla^2\left[B_i\right]\\&=D_i\nabla^2\left[B_i\right]_T\end{aligned} \tag{10}$$

And

$$R_i=-k_i^+\left[Ca^{2+}\right]\left[B_i\right]+k_i^-\left(\left[B_i\right]_T-\left[B_i\right]\right) \tag{11}$$

where

$$\left[B_i\right]_T=\left[CaB_i\right]+\left[B_i\right] \tag{12}$$

Thus, $[B_i]_T$, profiles are initially uniform and there are no source or sinks for Ca^{2+} buffer, $[B_i]_T$ remains uniform for all times.[11, 12] Thus, the following equations are written for the diffusion of Ca^{2+},

$$\frac{\partial\left[Ca^{2+}\right]}{\partial t}=D_{Ca}\nabla^2\left[Ca^{2+}\right]+\sum_i R_i+J \tag{13}$$

$$\frac{\partial\left[B_i\right]}{\partial t}=D_i\nabla^2\left[B_i\right]+R_i \tag{14}$$

where

$$R_i=-k_i^+\left[Ca^{2+}\right]\left[B_i\right]+k_i^-\left(\left[B_i\right]_T-\left[B_i\right]\right) \tag{15}$$

Here both R_i & J have units of concentration per unit time. Considering simplification of equations (6) to (8) that come about when buffer parameters are in select regimes: the so called "excess buffer" approximation.

In the excess buffer approximation (EBA), equations (6) to (8) are simplified by assuming that the concentration of free Ca^{2+}buffer $[B_i]$, is high enough such that its loss is negligible. The EBA gets its name because this assumption of the unsaturability of Ca^{2+} buffer is likely to be valid when Ca^{2+} buffer is in excess. [11, 12]

The association and dissociation rate constants for the bimolecular association reaction between Ca^{2+} and buffer can be combined to obtain a dissociation constant, K_i.

$$K_i=k_i^-/k_i^+ \tag{16}$$

This dissociation constant of the buffer has units of μM and is the concentration of Ca^{2+} is necessary to cause 50% of the buffer to be in Ca^{2+} bound form. To show this consider the steady state of equations (6) to (8) in the absence of influx (J=0). Setting the left hand sides of equation (7) and (8) to zero gives,[11, 12]

$$\left[B_i\right]_\infty=\frac{K_i\left[B_i\right]_T}{K_i+\left[Ca^{2+}\right]_\infty} \tag{17}$$

and

$$\left[CaB_i\right]_\infty=\frac{\left[Ca^{2+}\right]_\infty\left[B_i\right]_T}{K_i+\left[Ca^{2+}\right]_\infty} \tag{18}$$

where $[Ca^{2+}]_\infty$ is the "background" or ambient free Ca^{2+} concentration. And $[B_i]_\infty$ and $[CaB_i]_\infty$ are the equilibrium concentrations of free and bound buffer with respect to index i. In these expression K_i is the dissociation rate constant of buffer i. Note that higher values for K_i imply that the buffer has a lower affinity for Ca^{2+} and is less easily saturated. In this case, the equation for the diffusion of Ca^{2+} becomes,

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$$\frac{\partial\left[Ca^{2+}\right]}{\partial t} = D_{Ca}\nabla^2\left[Ca^{2+}\right] - \sum_i k_i^+ \left[B_i\right]_\infty \left(\left[Ca^{2+}\right] - \left[Ca^{2+}\right]_\infty\right) \quad (19)$$

To complete a reaction – diffusion formulation for the buffered diffusion of Ca^{2+}, a particular geometry of simulation must be specified and equation (19) must supplement with boundary conditions. If Ca^{2+} is released from intracellular Ca^{2+}stores deep within a large cell (so that the plasma membrane is far away and doesn't influence the time course of the event), and the intracellular milieu is homogenous and isotropic, then it has cylindrical symmetry [6]. In this case the evolving profiles of Ca^{2+} and buffer will be a function of r and θ only. For a two dimensional steady state case the equation (19) in polar in absence of influx (J = 0) is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\left[Ca^{2+}\right]}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\left[Ca^{2+}\right]}{\partial\theta^2} - \frac{k^+\left[B\right]_\infty}{D_{Ca}}\left(\left[Ca^{2+}\right] - \left[Ca^{2+}\right]_\infty\right) = 0 \quad (20)$$

The reasonable boundary condition for this simulation is uniform background Ca^{2+} profile of $[Ca^{2+}]_\infty = 0.1\ \mu M$. It is required that buffer far from the source to remain in equilibrium with Ca^{2+} at all times. Thus the boundary condition on the boundary away from the source is given by [11, 12]

$$\lim_{r\to\infty,\theta\to 0}\left[Ca^{2+}\right] = \left[Ca^{2+}\right]_\infty \quad (21)$$

At the source, it is assumed that influx takes place and therefore the boundary condition is expressed as [11, 12]

$$\lim_{r\to\infty,\theta\to\pi}\left(-2\pi D_{Ca} r\frac{\partial\left[Ca^{2+}\right]}{\partial r}\right) = \sigma_{Ca} \quad (22)$$

We define an influx of free Ca^{2+} at the rate σ_{Ca} by Faraday's law, $\sigma_{Ca} = \frac{I_{Ca}}{zF}$ [11, 12]. Hence, the problem reduces to find the solution of equation (20) with respect to the boundary conditions (21) and (22). Here, $\left[Ca^{2+}\right]_\infty$ is the background calcium concentration, $\left[B\right]_\infty$ is the total buffer concentration, σ_{Ca} represents the flux. $\left[Ca^{2+}\right]$ tends to the background concentration 0.1 μM as r tends to ∞ and θ tends to π. But the domain taken here is not infinite but finite one. Here, the distance is taken required for $\left[Ca^{2+}\right]$ to attain background concentration 7.8 μm for the Cardiac Myocytes (i.e. radius of the Cardiac Myocytes) [5]. Now the finite element method is employed to solve Equation (20) with boundary conditions (21) and (22). [6, 7, 8]

Assuming that the cardiac myocytes of circular shape and it is divided into coaxial circular elements [11, 12], given in figure 1.

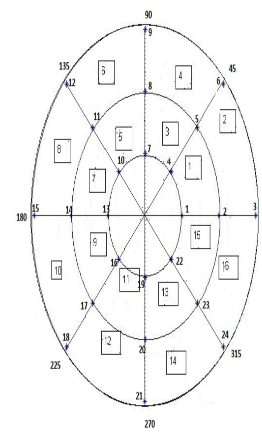

Figure 1: Finite element Discretization of circular cell

Here the square represents the number of elements and without square represents the nodal points where the nodal point 15 represents point source of calcium. The following table represents the element information.

Table 1: Element information

e	i	j	k	l
1	1	2	4	5
2	2	3	5	6
3	4	5	7	8
4	5	6	8	9
5	7	8	10	11
6	8	9	11	12
7	10	11	13	14
8	11	12	14	15
9	13	14	16	17
10	14	15	17	18
11	16	17	19	20
12	17	18	20	21
13	19	20	22	23
14	20	21	23	24
15	22	23	1	2
16	23	24	2	3

The discretize variational form of Equation (20) is given by

$$I^{(e)} = \frac{1}{2}\int_{r_i}^{r_j}\int_{\theta_i}^{\theta_j}\left[\left(\left(r\frac{\partial u^{(e)}}{\partial r}\right)^2 + \left(\frac{\partial u^{(e)}}{\partial \theta}\right)^2\right)\right]drd\theta$$
$$+\frac{1}{2}\int_{r_i}^{r_j}\int_{\theta_i}^{\theta_j}\left[\frac{k^+[B]_\infty}{D_{Ca}}r^2u^{(e)2} - \frac{2k^+[B]_\infty}{D_{Ca}}u_\infty u^{(e)}r^2\right]drd\theta$$
$$-\int_{\theta_i}^{\theta_j}\left[\frac{\sigma_{Ca}}{2\pi D_{Ca}}u^{(e)}\right]d\theta \qquad (23)$$

Here, 'u' is used in lieu of $\left[Ca^{2+}\right]$ for our convenience, e = 1, 2,…16.

The following bilinear shape function for the calcium concentration within in each element has been taken as [4, 11, 12].

$$u^{(e)} = C_1^{(e)} + C_2^{(e)}r + C_3^{(e)}\theta + C_4^{(e)}r\theta \qquad (24)$$

The thickness of each element is very small, therefore $u^{(e)}$ is assigned bilinear variation with respect to position as given by Equation (24).

In matrix form the equation (24) can be written as

$$u^{(e)} = P^T C^{(e)} \qquad (25)$$

where

$$P^T = [1 \quad r \quad \theta \quad r\theta] \quad and \quad C^{(e)} = \begin{bmatrix} C_1^{(e)} \\ C_2^{(e)} \\ C_3^{(e)} \\ C_4^{(e)} \end{bmatrix}$$

Also

$$u_i^{(e)} = C_1^{(e)} + C_2^{(e)}r_i + C_3^{(e)}\theta_i + C_4^{(e)}r_i\theta_i \qquad (26)$$
$$u_j^{(e)} = C_1^{(e)} + C_2^{(e)}r_j + C_3^{(e)}\theta_j + C_4^{(e)}r_j\theta_j \qquad (27)$$
$$u_k^{(e)} = C_1^{(e)} + C_2^{(e)}r_k + C_3^{(e)}\theta_k + C_4^{(e)}r_k\theta_k \qquad (28)$$
$$u_l^{(e)} = C_1^{(e)} + C_2^{(e)}r_l + C_3^{(e)}\theta_l + C_4^{(e)}r_l\theta_l \qquad (29)$$

Using Equations (26)-(29) we get

$$\overline{u}^{(e)} = P^{(e)}C^{(e)} \qquad (30)$$

where

$$P^{(e)} = \begin{bmatrix} 1 & r_i & \theta_i & r_i\theta_i \\ 1 & r_j & \theta_j & r_j\theta_j \\ 1 & r_k & \theta_k & r_k\theta_k \\ 1 & r_l & \theta_l & r_l\theta_l \end{bmatrix} \quad and \quad \overline{u}^{(e)} = \begin{bmatrix} u_i^{(e)} \\ u_j^{(e)} \\ u_k^{(e)} \\ u_l^{(e)} \end{bmatrix}$$

From Equation (25) and (30) we get

$$u^{(e)} = P^T R^{(e)}\overline{u}^{(e)} \qquad (31)$$

where $R^{(e)} = P^{(e)-1}$

Now the integral given in Equation (23) can also be written as,

$$I^{(e)} = \frac{1}{2}\int_{r_i}^{r_j}\int_{\theta_i}^{\theta_k}\left[\left(r\,\mathrm{P}_r^T\,R^{(e)}\overline{u}^{(e)}\right)^2 + \left(P_\theta^T R^{(e)}\overline{u}^{(e)}\right)^2\right]drd\theta$$
$$+\frac{1}{2}\int_{r_i}^{r_j}\int_{\theta_i}^{\theta_k}\left[\frac{k^+[B]_\infty}{D_{Ca}}r\left(\mathrm{P}^T\,R^{(e)}\overline{u}^{(e)}\right)^2\right]drd\theta$$
$$-\frac{1}{2}\int_{r_i}^{r_j}\int_{\theta_i}^{\theta_k}\left[\frac{2k^+[B]_\infty}{D_{Ca}}u_\infty r\left(\mathrm{P}^T\,R^{(e)}\overline{u}^{(e)}\right)\right]drd\theta$$
$$-\int_{\theta_i}^{\theta_k}\left[\frac{\sigma_{Ca}}{2\pi D_{Ca}}\overline{u}^{(e)}\right]d\theta \qquad (32)$$

Now $I^{(e)}$ is minimized with respect to $\overline{u}^{(e)}$

$\frac{dI^{(e)}}{d\overline{u}^{(e)}} = 0$, where

$\overline{u}^{(e)} = \left[u_i \quad u_j \quad u_k \quad u_l\right]^T, e = (1, 2, \ldots 16)$

$$\frac{dI}{d\overline{u}^{(e)}} = \sum_{e=1}^{N}\bar{M}^{(e)}\frac{dI^{(e)}}{d\overline{u}^{(e)}}\bar{M}^{(e)T}$$

$$\bar{M}^{(e)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 \end{bmatrix}_{(24\times 4)} \begin{matrix} \\ (i^{th}\ row) \\ (j^{th}\ row) \\ (k^{th}\ row) \\ (l^{th}\ row) \\ \\ \\ \end{matrix} \; and \; I = \sum_{e=1}^{16} I^{(e)}$$

This leads to following system of linear algebraic equations

$$[K]_{(24\times 24)}[\overline{u}]_{(24\times 1)} = [F]_{(24\times 1)} \qquad (33)$$

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Here, $\overline{u} = \left[u_1 \quad u_2 \quad \bullet \quad \bullet \quad \bullet \quad u_{24} \right]^T$, K is characteristic matrix and F is characteristic vector. Gaussian Elimination method is employed to solve the system (33).

III RESULTS AND DISCUSSION

A computer program in MATLAB 7.10.0.499 is developed to find numerical solution to the entire problem. The time taken for simulation is nearly 8.81 seconds on Core (TM) i 5-520M 330 @ 2.40 GHz processing speed and 4 GB memory. [6, 7, 8] To find the solution of equation (3.2.10) the biophysical parameters are taken from the literature as given in Table 2,

R	Radius of the cell	7.8 μm
I_{Ca}	Amplitude of elemental Ca^{2+} release	1 p A
F	Faraday's constant	96500 C/mol
Z	Valence of Ca^{2+} ion	2
D_{Ca}	Diffusion coefficient of free Ca^{2+} in cytosol	250 $\mu m^2 / s$
$[B_i]_T$	Total concentration for each Ca^{2+} buffer (Troponin C)	70 μM
k_i^+	Association rate constant for Ca^{2+} binding (Troponin C)	39 $\mu M^{-1} S^{-1}$
k_i^-	Dissociation rate constant for Ca^{2+} binding (Troponin C)	20 S^{-1}
K_i	Dissociation constant (Troponin C) $= \frac{k_i^-}{k_i^+}$,	0.51 μM
$[Ca]_\infty$	Intracellular free Ca^{2+} concentration at rest	0.1 μM

Table 2: Numerical values of biophysical parameters [5]

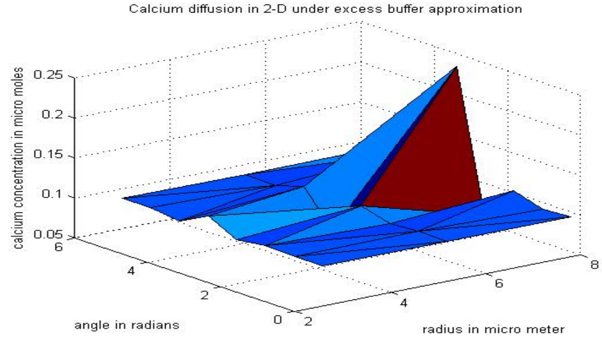

Figure 2: Two dimensional calcium diffusion under excess buffer approximation

Figure 2 represents the variation in Ca^{2+} concentration along angular and radial direction. We observe that maximum calcium concentration is 0.24 μM at a node (r=7.8, θ= π) i.e. source and it decreases along the angle as we move away from the source and it achieves its background concentration 0.1 μM at the other end i.e. at θ = 0. Initially from θ = π to θ = 3π/2 and θ = π to θ = π/2 the concentration falls rapidly due to buffering and slowly then after converges to its background concentration 0.1 μ M as θ approaches to 0 and 2π along the radial direction.

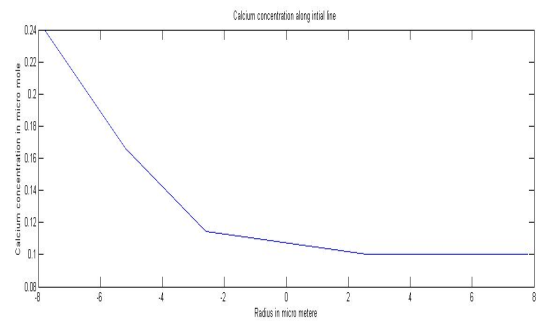

Figure 3: Calcium concentration along initial line

Figure 3 represents the variation in Ca^{2+} concentration along radial direction for initial line. We observe that maximum calcium concentration is 0.24 μM at r = -7.8 μm i.e. source and it decreases along the radius as we move away from the source and it achieves its background concentration 0.1 μM at the other end i.e. at r = 7.8 μm. Initially from r = -7.8 μm to r = -2.6 μm the concentration falls rapidly due to buffering and slowly then after converges to its background concentration 0.1 μ M as r approaches to -2.6 μm to 2.6 μm. Then it achieves its background concentration 0.1 μM and remains 0.1 μ M from r= 2.6 μm to 7.8 μm.

Conclusion: It is observe that due to excess buffer, the buffering activity has significant effect in calcium diffusion in cardiac myocytes give us better central regions little away from the source. The coaxial elements used here give us better approximations. The finite element method is quite flexible and powerful in dealing such problems and gives useful results in two dimensions.

IV REFERENCES

[1]. Backx P. H., De Tonb P. P., Jurjen H. K. ,Van Deen, Barbara J. M., Mulder, and Henke. D. J., "A Model of Propagating Calcium-induced Calcium Release Mediated by Calcium diffusion", The Journal of General Physiology, Volume 93, 1989, pp 963-977.

[2]. Luo and Rudy, "A Dynamic Model of the Cardiac Ventricular Action Potential. I Simulations of Ionic Currents and Concentration Changes", 1994a Circular Research, Volume 74 pp 1071-1096.

[3]. Luo and Rudy, "A Dynamic model of the Cardiac Ventricular action potential. II After depolarization, triggered activity and potentiation", 1994b Circular Research, Volume 74 pp 1097-1113.

[4]. MatveevnV., Zuker R.S. and Sherman, " Facilitation through Buffer Saturation: Constraints on Endogenous Properties" Biophys Journal, Volume 86, 2004, 2691-2709.

[5]. Michailova A. DelPrincipe F., Egger M. and Niggli E., "Spatiotemporal Features of Ca2+ Buffering and Diffusion in Atrail Cardiac Myocytes with Inhibited Sarcoplasmic Reticulum", Biophysical Journal, Volume 83, December 2002, 3134-3151.

[6]. Rao S.S., "The Finite Element Method in Engineering",2004, Elsevier Science & Technology Books, ISBN 0750678283

[7]. Reddy J. N., "An Introduction to the Finite Element Method", 2005, Tata McGraw-Hill Publication Company Limited, ISBN 0-07-053084-X.

[8]. Seshu P., "Textbook of Finite Element Analysis", 2003, Prentice –Hall of India Private Limited, ISBN 81-2003-2315-7.

[9]. Shannon T. R., Wang F., Puglisi J., Weber C. and Bers D. M., "A Mathematical Treatment of Integrated Ca Dynamics within the Ventricular Myocytes", Biophysical Journal, Volume 87, 2004, pp 3351-3371.

[10]. Smith G. D., Keizer J. E., Stern M. D., Lederer W. J. and Cheng H.,"A Simple Numerical Model of Calcium Spark Formation and Detection in Cardiac Myocytes", Biophysical Journal, Volume 75, 1998, pp 15-32.

[11]. Tiwari S. and Pardasani K. R., "Finite Element Model to study the Two Dimensional Unsteady state Cytosolic Calcium Diffusion in Presence of Excess Buffers", International Journal of Applied Mathematics, 2010, 40 3 01.

[12]. Tripathi A. and Adalkha N., "Two Dimensional Coaxial Circular Elements in FEM to study Calcium Diffusion in Neuron Cells", Applied Mathematics Science, Volume 06, 2012-466.

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Literary Studies and Communication Skills: A Possible Integration

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***Abstract*-- This paper tries to suggest certain ways in which the two traditionally segregated fields--literary studies and communication skills--can somewhat be integrated. The text with which the paper deals is Robert Browning's "My Last Duchess," a poem that lends itself to both literary studies and the study of communication skills. The object of the article is to bridge the gap between the two distinct fields of study by highlighting some aspects of the two fields which are socially relevant.**

***Index Terms*--Communication, communication skills, literary studies, literature.**

I

The phrase 'literary studies' refers to the study of literary texts which is systematic and specialised. Principles of hermeneutics--the origins of literary studies--are predicated upon the interpretation of biblical texts. The Reformation that took place in the Renaissance England was instrumental in the propagation of the idea that one had a personal relation with God, and that the relation can be comprehended without the help of any mediator. By the same token, people began to produce treatises on the interpretation of scriptures. Simultaneously, there emerged a growing collective concern for law and governance which enabled hermeneutics to enter the domain of jurisprudence. As Mario Klarer writes, the "approaches and methodologies associated with both (the exegesis of the Bible and the interpretation of legal texts) have always indirectly influenced literary studies" (76). With the development of the cult of 'Genius,' the quality of having spiritual insight or extraordinary mental acuity, in the Romantic period, hermeneutics entered the field of literature. That was the age when literary figures, especially poets, considered themselves as "unacknowledged legislators of the world;" hence the importance of the interpretation of literature in that period. From that time to the present, literary studies have undergone several changes. The famous categorisation of literary criticism done by M.H. Abrams in *The Mirror and the Lamp* (1953) (facilitates our understanding of the evolution of literary studies vis-à-vis English literature as well as literatures in English. Each of the four coordinates of English literary practice--author, reader, text and universe--have gained prominence one after another in accordance with the development of literary studies. In the twentieth century literary studies underwent radical changes with the advent of literary theory, a systematic speculative undertaking often resembling philosophy which concerns itself with the methods used in literary criticism and asks some fundamental questions like what is the cause of a particular literary production. It functions "as the theoretical and philosophical consciousness of textual studies, constantly reflecting on its own development and methodology" (Klarer 77). It is undeniable that with the rise of the pan-European literary theories such as Marxism, feminism, and new historicism English literary studies gained a 'socio-materialist' dimension. With its merging with culture studies and incorporation of postcolonialism, literary studies have become eclectic and diversified. Close reading championed by New Critics and deconstructionists is not the only objective of literary studies. Today, the entire field of study lays tremendous stress on the interaction between the literary text and history; it considers literary production as a socio-cultural signifying practice.

As David H. Fowler and Lois Josephs Fowler contend, the purpose of education in Europe and America, until the nineteenth century, was "primarily moral and cultural rather than utilitarian" (43). The rise of literary studies was predicated on the assumption that "learning was for the favored few, drawn from the gentry and the aspiring middle classes, rather than the many" (44). With the tremendous development of science and technology, rapid industrialisation, proliferation of political democracy and expansion of the middle class, there came the need for utilitarian education, that is to say, an education which would enable engineers and technicians to gain certain skills of communication required for their profession. English learning gradually had two ends: one was the pursuit of the Arnoldian 'high culture,' a specialised field of study, namely literary studies; the other was based on rather utilitarian principles catering to the needs of the people of science and technology, and hence dealt with the art of communication in engineering colleges and other technical institutes.

'Communication skills' denotes the socio-cultural/behavioural expertise which can be learned systematically. The field of study conceptualises certain social behaviour as skilled performances. However, "it was not until 1960 that the notion of communication as a form of skilled activity was first suggested" (Hargie 4). This consideration has gained prominence in the field of scientific and technological education. In other words, scientists and engineers have to master certain communication skills in order to disseminate their knowledge. They need to understand the value of communication skills and hence have to learn them systematically. It is to be noted that in India communication skills in English have been incorporated in the syllabi of students of scientific and technological institutes.

In India, it is undeniable that the dominance of English studies is a result of a protracted colonial history. The introduction and perpetuation of English language and literature studies was a part of the strategy of dominance of the British rulers in India. It was a discursive (cultural) practice which aptly bolstered the material (socio-politico-economic) dominance of the colonial rulers over the natives; thereby establishing, what Antonio Gramsci calls, 'hegemony' (Vishwanathan). It is to be noted that Indian writers and academicians have been trying to resolve the conflict between 'abrogation' and 'appropriation,' between the rejection of English as the language and literature of the coloniser, and the appropriation of English for postcolonial ends. Today, English language and literature have become a part and parcel of the cultural consciousness and material existence of many Indian people. As Raja Rao argues, we must approach English not as a language which is alien to us; as most of the postcolonial theorists, writers and critics maintain, we must not abrogate English but appropriate it for our own ends (Gandhi 151). There is no need to mimic the people of England and try speaking like them; we must advocate 'hybridity' and 'syncretism' while using the language. The present scenario in India is suggestive of the importance of English in both literary studies and communication skills.

The aim of this paper is to suggest a few ways in which literary studies and the study of communication skills can intersect each other. The text that this paper deals with is Robert Browning's "My Last Duchess." The dramatic monologue has been a subject of several scholarly discussions. Here, I would try to discuss it in terms familiar to both the students of literature and communication skills.

II

In the dramatic monologue "My Last Duchess," we only listen to the megalomaniac Duke of Ferrara. The poem begins *in media res*: the duke is showing an emissary the picture of the last duchess, her deceased wife. She is no more than a painting now, a work of art. According to the duke, the duchess had a heart "too soon made glad/Too easily impressed; she liked whate'er/She looked on, and her looks went everywhere./… 'twas all one!" (Browning 1842). In terms of communication skills, we may argue that the duke's version indicates that there was an intrapersonal barrier in communication between him and the duchess. The duke is somewhat blinded by what is known as 'blocked categories' or prejudices: he misperceived the smile, a non-verbal, kinesic communicative action. He gave commands and "all smiles stopped together. There she stands /As if alive" (1842). These pithy statements betray the megalomania of the duke. As Joshua Adler contends,
"The works of art to which he draws his hearer's attention in the aesthetic frame are pointed out not so much for their intrinsic beauty as for the fact that they are evidence of their owner's fine connoisseurship and ability to commission the most skillful artists. This is complemented by the demonstration of his insistence that his authority shall not be trifled with: only he may draw aside the curtain covering his wife's portrait. And while he feels he has an uncanny gift for reading his companions' thoughts, they dare never, he thinks, ask the question whose answer might satisfy the curiosity aroused by depth and passion of the

sitter's glance. Similarly, the "rarity" at the end of the poem, the possession of which is an instance of his prowess as an art collector is obviously an emblem of his own condition--Neptune taming a sea-horse, a creature symbolic of vitality and freedom" (220).

It seems, the duke wanted to assume total control over the duchess. Smile is usually considered to be a spontaneous reaction. However, the duke wanted the duchess to smile in such a way that would always underscore his authority and power over her. The duke complains that with her smile his last wife could not differentiate between the presence and favour of the duke and those of any other person: "Oh sir, she smiled, no doubt, /Whene'er I passed her; but who passed without /Much the same smile?" (1842). The duke's fragile male ego prevents any effective communication to take place between him and the duchess, and the consequences are terrible. He never tries to think from the perspective of the duchess. Instead, he expects from her only what he deems suitable for an ideal duchess; that is to say, the duke perceived his last wife in accordance with his notion of wifehood. Since the duchess did not conform to the standards of wifehood set by the duke, she had to die. Now having her transfixed as a portrait, the duke can enjoy total control over her. At the same time the duke perhaps likes to have the authoritative position where the duchess is only accessible to the envoy--the interlocutor of the dramatic monologue--as well as to the reader through the image that the duke deems suitable: "since none puts by/ The curtain I have drawn for you, but I" (1842). We can never get the version of the duchess, what she thought about the duke; instead we have to rely on the duke completely.

It might be argued that by narrating the story of 'disobedience' and 'fall' of the last duchess, the duke is trying to warn the envoy--who has probably come for a prospective marriage of the duke--about the behaviour of the future duchess: the future duchess might face the same consequence if she does not conform to the ideals of duchy set by the duke. Therefore, there is anticipation in our minds that the same intrapersonal barrier will occur in future as well. Analysing the source of this intrapersonal barrier from feminist perspective, one can contend that the duke's megalomania is symptomatic of the patriarchal ideological construct prevalent in the Victorian age. As Lineszy-Overton argues, "The Duchess's smile was a symbol of her connection with the outside of her marriage. The Duchess was able to communicate and bond with others through her smile. Likewise, Victorian women will be able to communicate and bound [bond] with others through education. One of the reasons Victorian men oppose educating women is that education frees women from being dominated by men. When it comes to the improvement of women's lives, Victorian men are interested only in what will benefit men. Consequently it is easier to control an uneducated wife, than one that is intellectually equal to her husband" (4). Here Lineszy-Overton draws upon Mary Wollstonecraft's *A Vindication of the Rights of Woman: with Strictures on Political and Moral Subjects* (1792). The treatise of Wollstonecraft is regarded as a strong protest against the contemporary discursive formations which objectified women. Being an advocate of the Enlightenment principles which championed the idea that education is instrumental in human emancipation from the dogmatic and authoritative Wollstonecraft--a prominent figure of first wave feminism--believed that women have right to education and must be a 'companion' to their husbands rather than mere 'wives.' It is to be noted that this claim of Wollstonecraft does not seem to emphasise the liberal feminist idea of equal social rights of men and women; but she was surely against the sheer objectification of women by patriarchy.

As Keval Kumar argues in *Mass Communication in India* (1994), right to communicate is one of the fundamental rights of the individual(6). By her smile the last duchess was exercising nothing but her fundamental right to communicate. On the other hand, the duke's "defensiveness--" a psychological barrier to communication germinating from "man's most compelling needs," that is, "to justify himself--" is evident in his words. For instance, noticeable is the way in which he feels the urge to explain and justify his "commands" that stopped "all smiles:" "How such a glance came there; so, not the first/ Are you to turn and ask thus. Sir, 'twas not/ Her husband's presence only, called that spot/ Of joy into the Duchess' cheek" (1842). After this, the duke enumerates instances of her disobedience in order to justify his ruthlessness and megalomania. His egotism was disturbed by the duchess' smile probably because it was an assertion of her subjecthood, an exercise of her fundamental right, that is, right to communicate.

III

In conclusion, it might be said that neither communication nor literature is asocial. Communication is not a synthetic process in which isolated individuals exchange discreet information (Kumar 8). It is rather a "social and cultural 'togetherness'" (1). By the same token, one must assert that literature is not an isolated pursuit of aesthetic pleasure only. Rather, it is a historically conditioned signifying practice. Instead of segregating them we must integrate them in order to realise their social relevance. The paper studies a literary text and tries to bridge the gap between the study of communication skills and literary studies by showing some areas in the text that lends itself to the two fields of study.

I. References

[1] M. Klarer, *An Introduction to Literary Studies*, 2nd ed. Routledge: London, 2004.

[2] D. H. Fowler and L. J. Fowler, "Literary Studies and Communication Skills: Separation or Reconciliation?" *The English Journal*, vol. 73.3, pp. 43-48, Mar. 1984.

[3] G. Vishwanathan, "The Beginnings of Literary Study in British India," *Oxford Literary Review*, vol. 9.1&2, pp. 431-437, 1987.

[4] G. Vishwanathan, *Masks of Conquest: Literary Studies and British Rule in India*, Faber and Faber: London, 1989.

[5] L. Gandhi, *Postcolonial Theory: A Critical Introduction*, Oxford University Press: New Delhi, 2008.

[6] O. Hargie, *Skilled Interpersonal Communication: Research, Theory and Practice*, 5th ed. Routledge: London, 2011.

[7] R. Browning, *The Norton Anthology of Poetry*, 5th ed. Ed. M. Ferguson, M. J. Salter, and J. Stallworthy. New York: Norton, 2005.

[8] J. Adler, "Structure and Meaning in Browning's 'My Last Duchess,'" *Victorian Poetry*, Vol. 15.3, pp. 219-227, 1977.

[9] K. J. Kumar, *Mass Communication in India*, Jaico Publishing House: Mumbai, 1994.

Fitting Inclusive Growth Model for Indian Economy

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Abstract--

The genesis of inclusive growth as an alternate development strategy can be traced in the order of Washington Consensus, Pro Poor Growth Strategy) and UNDP sponsored MDGs There is no universal definition of inclusive growth. However the scanning of existing literature reveal that there are relatively few but well founded studies, reports and publications on inclusive growth which are the knowledge products of World Bank, IMF, International Policy Centre for Inclusive Growth – IPCG, an initiative by UNDP, Asian Development Bank (ADB) etc. Inclusive growth emphasizes ensuring that the economic growth generates not only new opportunities but also give equal access to these opportunities to all particularly to the poor the maximum possible extent. This paper explores alternate economic growth models through a theoretical approach and then attempt to fit the inclusive growth model for Indian Economy
Key Words: Inclusive Growth, Alternate Growth Models, Pro Poor Growth.

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I - Introduction

Economic growth means an increase in real per capita income. If the economic growth causes inequitable distribution of opportunities (higher Gini coefficient) then it results in imbalanced development. Economic growth cannot be sensibly treated as an end in itself. Development has to be more concerned with enhancing the lives we lead and the freedoms we enjoy [4]. Economic growth alone does not guarantee the human development. Traditionally development means the capacity of a national economy whose initial economic condition is more or less static for a long time to generate and sustain an annual increase in its gross national income at a rate of 4% to 7% or more. A common alternative economic index of development has been the use of the rates of growth of income per capita to take into account the ability of a national to expand its output at a rate faster than the growth rate of its population, Levels and rates of growth of real per capita GNI (monetary growth of GNI per capita minus the rate of inflation) are normally used to measure the overall economic well being of a population – how much of real goods and services are available to the average citizen for consumption and investment [23]. Development has occurred when there has been improvement in basic needs, when economic progress has contributed to a greater sense of self esteem for the country and individuals within it and when material advancement expanded people's entitlements, capabilities and freedoms [2].

II. Research Objectives

The research objectives of this paper are as follows.

1. *To develop an analytical framework for the proposed research which will enable the state/country to assess the inclusive growth on any given period of time.*
2. *To explore alternate economic models and fit the best model which can capture the inclusive growth dynamics*
3. *To study whether there exist any correlation between pro poor growth and inclusive growth*
4. *To recommend an appropriate policy mix for achieving the goal of inclusive economic growth*

III. Research Questions

This paper would like to answer the following research questions.

Research Question #1: Is our pattern of growth fair and equitable i.e. inclusive?

Research Question #2:Can we change our underlying theories and models in use and begin to explore alternate growth models

Research Question #3:How to measure the inclusive growth in an economy? I.e. How to capture the entire relevant socio-economic and human development variable to diagnose the inclusive economic growths achieved by an economy?

Research Question #4: How to differentiate the pro-poor growth from inclusive growth? i.e. why the poor are unable to exploit growth promoting opportunities for investment in human capital?

Research Question #5: Is there a correlation between regional disparities and inclusive growth? I.e. whether the high level initial inequality impacts the inclusive growth? Does the geographical and sect oral pattern of economic growth impact the inclusive growth objective?

Research Question #6: What is the appropriate policy-mix for achieving inclusive in transition economies like India? I.e. how to quantify the trade-offs between alternative policies for promoting inclusive growth embracing both redistributive social policies and alternative growth strategies?

Research Question #7: How to strengthen local, community based collaborative governance?

Research Question #8: How to achieve sustainable growth given environmental constraints?

IV. Scanning of Existing Literature

Most of the studies on inclusive growth point to the economic dimension – the sustainable and equitable growth - key to achieve inclusive growth. They also pointed out that the growth should not only be equitable but also should be broad based across the sectors and regions. An enabling factor which drives inclusive growth and poverty reduction is the quality of infrastructure, particularly the rural infrastructure since large part of the poor people live in rural areas which are deprived of the spillover effect of huge public investment which are taking place in the urban areas [15]. Quality infrastructure provides the conducive business environment and investment climate for domestic and foreign direct investment. Therefore focus should be given on rural electrification, transport, communications and water supply. Inclusive Growth should focus on the importance of sustainable agricultural growth.[30]. Rural infrastructure contributes in providing access to markets and basic public utility services which ultimately impact the rural economic growth and employment opportunities and income generation [28]. It is abundantly clear that for the realization of laudable goal of inclusive growth in the country the problem of poverty needs to be proficiently tackled.

ADB's Poverty Reduction Strategy focused on the three pillars – pro-poor sustainable economic growth, social development and good governance and all these are in line with the MDGs. Several studies of Ali [18], Ali and Son [17], Ali and Zhuang [16] confirm the important role of promoting sustainable environmental growth to inclusive growth. .Lin [23] argued that continuous flow of technology and industrial innovation are key to sustained growth of any country. ADB studies [18] also calls for ensuring a level playing field to ensure equal opportunities to all particularly to poor the maximum possible extent. Ali and Zhuang [16] recommended that governments address institutional weaknesses and maintain the rule of law and needs to invest heavily in physical infrastructure and human capital. Ali and Son [17] noted that strengthening capabilities in the form of human capital supports inclusive growth. They call for providing social safety nets. Fernando [28] recognized that community based organizations, civil society organizations and non government organizations (NGOs) significantly contributes to promote equity and inclusiveness through participation in the allocation of resources for rural development, vigilance in the misuse of funds to prevent corruption and promote accountability and transparency and provisions of access to public services

Robust growth has reduced the number of people in developing regions living on less than 1.25 USD a day from 1.8b in 1990 to 1.4b in 2005 and poverty rate dropped from 46% to 27% [35]. The MDG targets should be achieved to fulfill the promise of Millennium Declaration for better world [9].[33],[34]

The Report of Tendulkar Committee is a significant step forward in the literature and policy debate on poverty in India [37] . The 1979 poverty line has endured not only on account of persistence of few statisticians but because government resist attempt at creating new rights and activists will not accept anything less. Therefore the last accepted truce continues. At a more practical level the N.C.Saxena Committee [27] set up for identifying beneficiaries through the latest BPL Census has presented a multi dimensional and many splendorous menu [37] The Tendulkar Committee [13] has moved over from a calorie determined poverty line to a food expenditure determined poverty line. The Report [13] has a concept of inclusive growth wherein the state does not take on itself such pro poor responsibilities but provides for a concept of income supplements for private expenditures for them.

Tang [33] identified the following challenges to inclusive growth: poverty threshold needs updating, targeting failures, insufficient financing, independent financial sector, inadequate attention to capacity development, gaps in urban poverty programs.

V. An Inclusive Growth Analytical Framework

The world bank (2009) research recommends an analytical framework for further exploring the theoretical base of inclusive growth model.Accordingly the economic growth should cause productive employment which should be capable of reducing income poverty. The World Bank study [11] designed the following framework in which the economic growth is considered as an engine of inclusive growth. However the Economic Growth should produce Productive Employment which can generate income capable of poverty reduction.

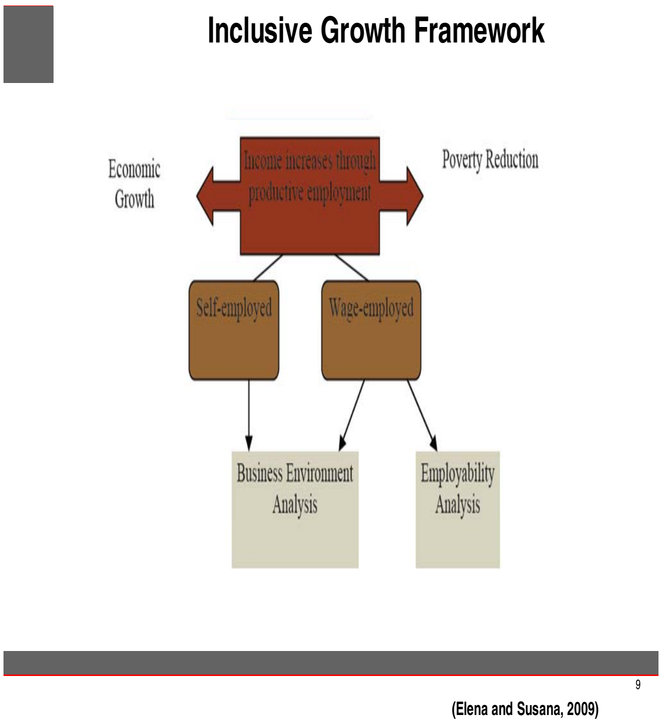

Figure - 1

India is not unique. High and rising spatial disparities are a feature of many developing countries. In Peru, for example, the incidence of income poverty in districts at sea level is three quarters of that in mountain districts. In China rural per capita income in Shanghai province is more than five times that in Guizhou province.[33]. Moreover, Chinese regional inequality increased throughout the 1990s and early 2000s, reaching an all time historical high. However many of them formed inappropriate strategy and impeded these opportunities to realize this growth potential [23] He has made a valiant effort in differentiating the CAD (Comparative Advantage Defying) strategy with CAF (Comparative Advantage Following) strategy and concluded that the government should encourage the firms to enter the industries for which the country has comparative advantages and to adopt production technologies that will make these firms viable. Economies under CAF will import what are not their comparative advantages and export what are their comparative advantages. Thus economies under CAF will accumulate more capital, faster upgrading of endowment structure, better income distribution, dynamic growth, better job opportunities to the poor and growth with equity [23].

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The Washington Consensus

The WC emerged in the early 1980s as a dramatic right-wing reaction against the perceived weaknesses of the pre-WC developmental consensus. Rhetorically, the WC involved a heavy attachment to a universality neo-liberal ideology, with absolute commitment to the free market and the presumption of the state as a source of both inefficiency and corruption, not least through rent-seeking (for a clear statement, see Krueger, 1974). At the level of scholarship, the WC suppressed the old development economics as a separate and respected field and instead imposed rigid adherence to the deductive and formal methods of neoclassical economics that were thought to be equally and directly applicable for analysis of the problems of poor countries [22]

Elements of Washington Consensus:

The WC comprised four elements. First is the hegemony of modern neoclassical theory within development economics. In general, the neoclassical theory assumes that the market is efficient and the state is inefficient. It naturally follows from this assumption that the market rather than the state should address such economic problems of development as industrial growth, international competitiveness and employment creation. Unquestioned belief in the neoclassical theory also leads to the assumption that capital mobility and the relentless advance of "globalization" is good for the world economy and all individual economies. Although these policies offer the possibility of rapid growth by attracting foreign capital, this can be achieved only if domestic policies conform to the interests of the (financial) markets—otherwise capital will be driven elsewhere. Finally, given the priority attached to monetary policies over fiscal policies, interest rates became the most important economic policy tool. It was believed that "correct" interest rates could deliver balance of payments equilibrium, low inflation, sustainable levels of consumption and investment, improved allocation of resources and, therefore, high long-run growth rates.

Second, for the pre-WC, the main reason why poor countries remain poor is their lack of capital (machines, infrastructure and money), and development is a process of systemic transformation through modernization and industrialization, driven by domestic consumption and domestically-financed capital accumulation. In contrast, in view of the WC, countries are poor because of misconceived state intervention, corruption, inefficiency and misguided economic incentives. According to WC, development is the inevitable outcome of a set of "appropriate" incentives and neoclassical economic policies, including fiscal restraint, privatization, the abolition of government intervention in prices, labor market "flexibility", and trade, financial, and capital account liberalization. There is little specification of what the end-state would look like but, presumably, all countries would eventually approach an idealized version of the United States.

Third, the WC emphasis on the virtues of the market was supported by the neo-Austrians associated with Friedrich von Hayek and the general equilibrium theory of mainstream economics [8]. . Despite the libertarian streak associated with these theories, even the most ardent supporter of freedom of the individual in general, and through the market in particular, agrees that these freedoms can be guaranteed only through state provision of, and coercion for, a core set of functions and institutions. These range from fiscal and monetary policies to law and order and property rights, and includes military intervention to secure the "market economy" when this becomes necessary. Not surprisingly, then, WC policies are often associated with authoritarianism, while the WC declarations of support for political democracy are hedged and conditional in practice While the WC claimed to be leaving as much as possible to the market, in practice it encouraged state intervention on a discretionary basis, and directed to systematic promotion of a globalize and heavily- financial capitalism.

Fourth, under the WC the World Bank set the agenda for the study of development, with the Bank and the IMF imposing the standards of orthodoxy within development economics, and enforcing the relevant policies through conditionality imposed on poor countries facing balance of payments, fiscal or financial crises.

The Pro Poor Policy Debate:

In the late 1990s the mainstream was compelled to admit that poverty reduction and redistribution were not spontaneous by-products of growth, the correction of macroeconomic imbalances, or improvements in macroeconomic policies and governance. Instead, poverty has to be addressed directly through a dedicated set of economic and social policy tools.

Growth process is called distribution neutral if the growth incidence curve is perfectly flat in such a way that all percentiles grow at the same rate, leaving inequality unchanged. The distributional change is pro poor if the redistribution reduces poverty sharply. Therefore the rate of pro poor growth is equal to the distributional correction multiplied by ordinary growth rate [26]

For Kakwani, pro-poor growth (PPG) is defined by the *increase in the income share of the poor* (alternatively, in PPG, the incomes of the poor grow faster than those of the non-poor, in which case poverty falls faster than it would if all incomes had grown at the same rat[29].. In contrast, Ravallion focused on the *absolute improvement of the living standards of the poor*, regardless of changes in inequality. Typically, Ravallion stressed the pro-poor implications of growth in China because it reduced absolute poverty, regardless of worsening inequality in the country[32].. While Kakwani rejected Ravallion's definition of PPG because it is too elastic and can potentially include most growth processes in history, Ravallion criticized Kakwani for the alleged inconsistency of his definition of PPG

"growth-enhancing policies and institutions tend to benefit the poor—and everyone else in society—equi-proportionately' [25]

In other words, while the impact of targeted interventions is both uncertain and weak, growth can *certainly* improve the welfare of the poor. Consequently, attempts to shift the income distribution are largely a diversion, and conventional policies [1]

The search for a general relationship between growth and equity has highlighted the implications of the two competing definitions of PPG commonly found in the literature. If PPG is defined as *growth that promotes equity*, equity becomes the key principle for the selection of economic policies, and only those policies which directly promote equity are "pro-poor". Conversely, if PPG is defined as *growth that improves the absolute condition of the poor*, PPG includes all non-perverse types of growth, and any poverty-alleviating policy is "pro-poor". In this case, equity has only instrumental value: it is a tool which *may* be deployed *if* it increases the poverty-alleviating impact of a given set of economic policies [32]

The logical consequence of shifting the terms of the debate away from the *principle of equity* and towards the *goal of poverty reduction* is the resolution of the PPG debate in terms that are unfavorable for promotion of equity.[7]. If everyone agrees that elimination of poverty is the ultimate goal, and admits that growth helps to achieve it, they can disagree only about the combination of policies which maximizes the poverty-reducing impact of growth (and which may or may not include certain modalities of equity) [12].

VI. Exploring Alternate Models of Growth

This paper now turns to the alternate models of growth which are confronted by the policy makers today to make the India growth story inclusive. The three distinct growth models have been identified by Planning Commission, Government Of India[14]. They are as follows. .

1. The Flotilla Muddling Along Model
2. The Flotilla Falling Apart Model
3. The Flotilla Advance Model (Strong Inclusive Growth Model)

1.The Flotilla Muddling Along Model:

Under this model some reforms are undertaken and some critical reforms are left out. Since government did not address core governance issues the reforms are not effective. This model shows a centralized government with inefficient allocation of resources which ultimately impedes the outcome of various centralized programmers. The policy conflicts between inclusive economic growth and populist measures in the name of pro-poor growth like MNREGA(2005), farm-waiver scheme(2009), Education for All (2010), Food Security Bill (2013) and Land Acquisition Bill (2013) etc. remain unresolved. The much needed second generation economic reforms are stalled and many infrastructure projects are stalled. There exist judicial outreach due to political leadership deficit and governance deficit. The recent coal gate issue, mine issue and telecom issue pose greater danger to the democratically rooted country like India where Parliament is supreme, not judiciary.

2. Flotilla Falling Apart Model:

The law making bodies like Parliament is not functioning and therefore many reforms are stalled. Myopic political vision coupled with political interregnum brings governance to the brink of disaster. Different key policy taking Ministries/Departments are pulling on different directions. Civil Society protest movement like the Anti Corruption movement by Anna Hazare make us wonder whether laws can be made in the street. All these result in political logjam in which government can hardly function. Governance ceased to exist. As investment shrinks, the celebrated India's growth story is fast disappearing and the old Hindu Growth Rate (around 3.5% GDP) is looming over the policy makers as unemployment spreads and India fail to encash the demographic dividends as there exist no productive employment which result in income poverty. To add fuel to the fire, the tight monetary policy by the Central Bank to arrest the inflation backfired and there is no coordination between the monetary and fiscal policies which complicated the problems and widened the current account deficit. Populist measures in the name of pro poor growth have widened the fiscal deficit and government fails to adhere to the basics of FRBM Act. The depreciating Indian Rupee worsened the economy and brought us back to those memories of 1991 situations. The only thing that differentiate 1991 and 2013 is that today we have sound foreign exchange reserves which can meet the import bill obligations of around six to eight months unlike in 1991 when it was hardly enough to meet the import bill obligations of 14 days.

3.Flotilla Falling Apart Model (Strong Inclusive Growth Model)

In this model the government swung into actions on war foot and immediately resolves the issue of depreciating Indian Rupee against USD, decimating the current account deficit (CAD), restricting the fiscal deficit within the orbit of FRBM Act and with a functioning Parliament many pending and key reforms should go through the legislation process. Once economy is back to normal then the policy maker can focus on designing the model to achieve inclusive growth. The inclusive growth model will accommodate the three pillars – inclusion, governance and sustainability. Therefore the inclusive growth is not only pro poor growth but also make human development which ensures participation of the poor in the growth process. This is the best model which can be fitted into the system of Indian economy with the interest of millions who were alienated from the growth story and their issues will be properly addressed and social safety measures will be taken so that people can be guarded from any internal or external shocks. Decentralized governance and solutions along with a focus on opportunity based inclusion produces more sustainable strength – socially and economically [14].

Identifying the Variables – Building Blocks for Inclusive Growth Analytic Framework. The following variables are identified which capture the inclusive growth framework for can diagnose of economic progress of a nation.

1. Inclusive Economic Growth: Endogenous Variable
2. Economic Growth: Exogenous Variable
3. Productive Employment: Exogenous Variable
4. Income Poverty: Exogenous Variable
5. Income Inequality: Exogenous Variable
6. Human Development Variables: Exogenous Variables Health and Literacy
7. Economic Infrastructure Access to Safe Drinking Water, Electricity, Toilet food, puce houses, all weather good roads

Inclusive Growth Model: The following model has been developed by incorporating the above variables.

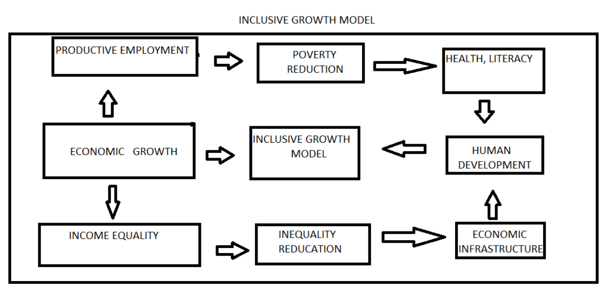

Figure – 2

The above model can well fit into the current Indian economic system the following policy recommendations can well be considered while converting the above model into policy implementation.

VII. Policy Recommendations

1. The government should incorporate a n appropriate policy mix that include promoting efficient and sustainable economic growth, ensuring a level political playing field and strengthening capacities for prudent social safety nets [16,17,28]

2. Strengthen institutional reforms like improving law and order, providing financial inclusion and delivering basic utility services like drinking water, sanitation, education and health [28]

3. Developing countries should adopt Comparative Advantages Following (CAF) strategy and concluded that the government should encourage the firms to enter the industries for which the country has comparative advantages and to adopt production technologies that will make these firms viable.[23]

4.Governments should strengthen capacities in formulating and implementing appropriate macro economic and social protection policies, accountability in fiscal administration, efficient delivery of public services and improved and quality governance [6],[7].

5. Government should provide an enabling environment for business by eliminating marketing distortions and institutional weaknesses [17].

6. Carefully designed redistributive strategy should also be part of an inclusive growth strategy [17]

7. An ago-ecological food system is a socially inclusive pathway to food security and has proved to be productive, resource-conserving and highly sustainable. Therefore agricultural policies should be directed to local level which can increase food security in a sustainable manner [19,20].

8. Efforts to promote equality and social inclusion can deliver beyond the micro scale and through technology transfer can create exponential effects both within and between countries. Social technology's capacity to value and create social capital also allows for practical and tangible ways for social policy, social well being and good social practice to be effectively leveraged reducing impacts both on and from the environment [19,20]

9. Within the delivery mechanism to address a particular MDG the UID database can be used to stratify information that is appropriate for each target group and it is only then that the power of identity can be combined with the power of technology to generate the power of information to be channeled into better service delivery, broader inclusion and greater accountability.[5].

10.Rangarajan [10] suggested the following policy-mix for achieving financial inclusion

- effecting improvement within the existing formal credit delivery mechanism
- evolving new models for effective outreach
- leveraging on technology based solution
- setting up of National Rural Financial Inclusion Plan
- use of financial literacy and credit counseling
- use of intermediaries as agents in Micro Finance

11.The Tendulkar Committee Report [13]) has moved over from a calorie determined line to a food expenditure determined poverty which has a concept of inclusive growth wherein the state does not take on itself pro poor responsibilities but provides for a concept of income supplements for private expenditures for them [37]

12Given the two-way causality and the nature of the trade-off in terms of time between economic and human development, the central institutions like Planning Commission and Finance Commission should only be guided by national priorities between economic growth and human development and the regional disparities are best left to the regions, otherwise inefficiencies may crop up and the national priorities may get sacrificed [31]

References:

[1] A.Kruegar (1974), The Political Economy of the rent seeking society" American Economic Review, Vol.64 pp.291-303

[2] A.P.Thirlwall(2011) "*Economics of Development"* P.20, Ninth Edition,

[3[Aashita Dawerta (2012) "Inclusive Growth: Significance of the Manufacturing Sector" The Asian Economic Review, Vol.54, No.1, April,2012

[4] Amartya Sen (1999) *Development as Freedom* p.14

[5] Anit Mukerjee (2012) Tackling Povery in India: The Role of the Unique ID Number – "Poverty in Focus" Number, 24,

[6] Asian Development Bank (1999) Poverty Reduction Strategy (PRS)

[7] Asian Development Bank 2007) Long Term Strategic Framework: Lessons from implementation (2001-2006), Manila.

[8] B.Fine and A.Saad Filho (2011), Neo-liberal development and its critics: In Politics of Neoliberalism" H Nobter ed, London, Rolctledge

[9] B.Moon (2010), "Millennium Development Report", UN

[10] C.Rangarajan ((2008), "Report of the Committee on Financial Inclusion"

[11] E . Ianchavichina and S. Lundistoram (2010), "Inclusive Growth Analytics: Framework Policy Research

Working Paper, *World Bank, EconomicPolicy and Debt Department, Economic Policy Division, WP: No.4851*

[12] Filho Saad Alfredo (2010) DESA Working Paper No.100, "Growth, Poverty and Inequality: From Washington Consensus to Inclusive Growth"

[13] Government of India (2009), Report of Expert Group to Review Methodology for Elimination of Poverty

[14] Government of India (2013), Planning Commission :"Scenarios: Shaping India's Future"

[15] I. Ali and X Yao (2004) Pro-poor inclusive growth for sustainable poverty reduction in developing Asia: the enabling role of infrastructure development, ERD Policy Brief series no.27, ADB

[16] I.Ali and H Son (2007) Defining and measuring inclusive growth: Asian prescriptions, ERD Policy brief series No.48, ADB, Manila

[17] I.Ali and J.. Zhuang (2007) Inclusive Growth towards a Prosperous Asia: Policy Implications" . ERD Working Paper No.97, ADB.

[18] Ifzal Ali (2007), "Inequality and the imperative for Inclusive Growth in Asia" – *Asian Development Review,* Vol.24, No.2, pp. 1-16,

[19] IPCIG (2012) Research Brief No.23, Bureau for Development Policy, UNDP and Government of Brazil

[20] IPCIG (2012),-Research Brief No.25, Bureau for Development Policy, UNDP and Government of Brazil IPCIG, Bureau for Development Policy, UNDP.

[21] J. Zhuang (2008), "Inclusive Growth toward a Harmonious Society in the People's Republic of China: Policy Implication", *Asian Development Review"* Vo.25, No.2, pp.22-33, 2008, June-Dec., 2008

[22] K.Jomo and B.Fine (2006), The New Development Economics After Washington Consensus, New Delhi, India, Tulika and London, Zed Press

[23] L..Y.Justin,(2004), "Development Strategies for Inclusive Growth in Developing Asia" ,"*Asian Development Review" ,* Vo.21, No.3,pp.1-27

[24] M.P.Todaro and S. C Smith (2012), "*Economic Development"* p.14, Tenth Edition.

[25] M.Ravallion (2004) , " Pro Poor Growth: A Primer" – World Bank Policy Research Working Paper, 3242

[26] M.Ravallion and S.Chen (1997) "What can new survey data tell us about recent changes in distribution and poverty, The World Bank Economic Review, Vol.11, No.2

[27] Ministry of Rural Development, Government of India (2009), *"The N.C.Saxena Committee Report"*

[28] N.Fernando,(2008), Rural Development outcomes and drivers: an overview and some lessons: EARD special studies, ADB, Manila

[29] N.Kakwani (2001), Pro-Poor Growth and Policies" , Manila, Philippines, Asian Development

[30] R.Bolt(2004) Accelerating agriculture and rural development for inclusive growth: policy implementation, ERD policy brief series no.29, ADB.

[31] R.Dholakia (2003) "Regional Disparity in Economic and Human Development in India"

[32] T.McKinley (2009), Revisiting the dynamics of growth, inequality and poverty reduction, Centre For Development Policy and Research, SOAS, Discussion Paper No.25/09

[33] Tang Min, (2008), "Inclusive Growth and the New Phase of Poverty Reduction in the People's Republic of China","*Asian Development Review"*, Vo.25, No.1, 2 pp.81-99, June-Dec.2008

[34] UN (2000),*"Millennium Development Goals – Millennium Declaration"*

[35] UN (2010),*"Millennium Development Goals (MDG) Report"*

[36] World Bank (2009). "*World Bank Development Report"*

[37] Y.K.Alag (2010), "The Poverty Debate in Perspective: Moving Forward with Tendulkar Committee", *Indian Journal of Human Development*, Vo.4,.

978-1-4673-1719-1/12/$31.00©2013IEEE

Numerical Optimal Control for Bilinear Hyperbolic PDEs

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***Abstract*--In this paper, we present a successive approximation scheme to solve finite-time optimal control problem for hyperbolic partial differential equations (PDEs) with additive, multiplicative and boundary controls. We propose an iterative scheme for control design. Finite difference scheme is applied to solve the hyperbolic equation. Control problem is solved using conjugate gradient method. A numerical simulation study shows the effectiveness of this approach.**

***Index Terms*--Bilinear systems, Hyperbolic PDEs, Conjugate gradient method, Optimal control.**

I. Introduction

ENGINEERING analysis requires prediction of outputs governed by partial differential equations. In many real-life applications of optimal control problems with constraints in the form of partial differential equations (PDEs), hyperbolic equations are involved which typically describe transport processes. Because of their nature being able to transport discontinuities of initial or boundary conditions into the domain on which the solution lives or even to develop discontinuities in the presence of smooth data, these problems constitute a severe challenge for both theory and numerics of PDE constrained optimization.

Dynamic stability due to parametric resonance is an important factor in structural dynamics. For example, it is believed that instability caused by parametric resonance is the reason for the famous Tacoma bridge collapse in 1940 [1]. Protection against this phenomenon can be achieved by a suitable control of the coefficients. Control in coefficients is an effective method in structures governed by elliptic problems [1]. However, the effect produced by bilinear control for hyperbolic problems is not studied regoursly [2]-[3]. Optimal control of systems governed by hyperbolic equations is of special importance for the active control of structural systems for which the equations of motion are generally expressed by hyperbolic differential equations. The field of structural control has been an active research area for a number of years. However, most of the studies in this area considered specific structures such as wings [4], beams [5] and plates [6]. Even though these studies provide solutions for many particular cases, the theoretical foundations of the subject aimed specifically at problems arising in structural mechanics have not received much attention. Theoretical studies such as the ones in [7], [8] considered the optimal control problems in abstract settings leaving a gap between the theory and applications. In particular, optimal control studies relating the theory directly to the solution method have been scarce. However, one such application is given in [9] for a structural vibration problem governed by a single hyperbolic equation. Another application is given herein for a vibration problem governed by a system of hyperbolic equations. The developed maximum principle [10] was used to construct explicit solutions for an optimal control problem involving a distributed parameter structure governed by a system of hyperbolic differential equations. In this paper, we study the numerical optimal control problem for a dynamic system governed by a hyperbolic equation with three types of controls together viz., bilinear controls, distributive controls and boundary controls.

The bilinear system is a kind of system where due to multiplicative controls system becomes nonlinear and it becomes more difficult to solve problems such as optimal control problems for these systems than those for linear systems. In literature, many researchers [11]–[20] have tried various methods to overcome the difficulties of solving control problems for bilinear parabolic systems. Theoretical studies such as the ones in [21]-[24] considered the optimal control problems in abstract settings for hyperbolic systems. In particular, optimal control studies relating the theory directly to the solution method have been scarce. However, such applications for time optimal control problem are given in [25], [26] for structural dynamic systems governed by a hyperbolic equation.

In this work an effort has been made to design an iterative control methodologies using conjugate gradient method. The paper is organized as follows: In Section II, we state the bilinear optimal control problem for hyperbolic PDEs. In section III, we convert the continuous optimal control problem into discrete control problem using finite difference approximation scheme. In Section IV, gradient based algorithm is developed for computing control functions. The simulation studies are presented in Section V. Section VI closes the paper by stating the conclusions.

II. Problem Statement

Let $\Omega_x = [0, L]$, $\Omega_t = [0, T]$ and $Q = \Omega_x \times \Omega_t$. Consider the wave equation:

$$u_{tt} = u_{xx} + a(x,t)u + b(x,t),\ (x,t) \in Q, \quad (1)$$

$$u_x(0,t) = g(t),\ u_x(L,t) = h(t),\ t \in \Omega_t, \quad (2)$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega_x, \quad (3)$$

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where $u \in H_0^1(\Omega_x)$ is the disturbance of position x at time t, $u_0,\ u_1 \in L^2(\Omega_x)$ and $a,\ b,\ g,\ h$ are control functions.

Let admissible control sets are

$$A_{ad} = \left\{ a(x,t) : 0 \le a(x,t) \in L^\infty(Q;\mathbb{R}) \right\}, \quad (4)$$

$$B_{ad} = \left\{ b(x,t) : 0 \le b(x,t) \in L^2(Q;\mathbb{R}) \right\}, \quad (5)$$

$$G_{ad} = \left\{ g(t) : 0 \le g(t) \in C^1(\Omega_t;\mathbb{R}) \right\}, \quad (6)$$

$$H_{ad} = \left\{ h(t) : 0 \le h(t) \in C^1(\Omega_t;\mathbb{R}) \right\}. \quad (7)$$

We now state an optimal control problem for the hyperbolic system (1)-(3) with the following cost functional

$$J(a,b,g,h) = \frac{1}{2}\int_Q (u-z)^2\, dQ + \frac{\alpha}{2}\int_{\Omega_t} a^2 dt + \frac{\beta}{2}\int_{\Omega_t} b^2 dt + \frac{\gamma}{2}\int_{\Omega_t} g^2 dt + \frac{\rho}{2}\int_{\Omega_t} h^2 dt, \quad (8)$$

where $\alpha,\ \beta,\ \gamma$ and ρ are positive weighing constants. $z \in L^2(Q)$ is a given target function. The cost functional combines the cost of control and the measure of closedness to the desired profile z. The optimal control problem is stated as follows: determine the optimal control functions $a^* \in A_{ad}$, $b^* \in B_{ad}$, $g^* \in G_{ad}$, $h^* \in H_{ad}$ such that

$$J(a^*,b^*,g^*,h^*) = \min_{a\in A_{ad},\ b\in B_{ad},\, g\in G_{ad},\, h\in H_{ad}} J(a,b,g,h), \quad (9)$$

subject to (1)-(3).

III. DISCRETE CONTROL PROBLEM

The control problem is discretized using finite difference approximation scheme. For the simplicity, we use a uniform grid and denote

$x_i = i\Delta x,\ t_j = j\Delta t,\ i = 0,1,\ldots,k,\ j = 0,1,\ldots m,$

$\Delta x = L/k,\ \Delta t = T/m,\ u_i^j = u(i\Delta x, j\Delta t),\ u_{di}^j = u_d(i\Delta x, j\Delta t),$

$a_i^j = a(i\Delta x, j\Delta t),\ b_i^j = b(i\Delta x, j\Delta t),\ u_{0i} = u_0(i\Delta x),\ u_{1i} = u_1(i\Delta x),$

$g^j = g(j\Delta t),\ h^j = h(j\Delta t),\ i = 0,1,\ldots,k,\ j = 0,1,\ldots m.$

We discretized the hyperbolic equation (1)-(3) with explicit forward Euler scheme in time, a centered scheme in space and use the simplest discretization in the vicinity of boundaries. We get the following discrete version of the continuous problem (1)-(3) with discrete version of cost functional (9) as follows:

$$\bar{J} = \frac{\Delta x \Delta t}{2}\sum_{i=0}^{k}\sum_{j=0}^{m}\mu_i^j (u_i^j - z_i^j)^2 + \frac{\Delta x \Delta t}{2}\sum_{i=0}^{k}\sum_{j=0}^{m}\left[\alpha\vartheta_i^j (a_i^j)^2 + \beta\kappa_i^j (b_i^j)^2\right] + \frac{\Delta t}{2}\sum_{j=0}^{m}\left[\gamma\lambda^j (g^j)^2 + \rho\sigma^j (h^j)^2\right], \quad (10)$$

$$u_i^j = \begin{cases} 2(1-\pi)u_i^{j-1} - u_i^{j-2} + \pi\left(u_{i+1}^{j-1} + u_{i-1}^{j-1}\right) & 1 \le i \le k-1, \\ +(\Delta t)^2 a_i^{j-1}u_i^{j-1} + (\Delta t)^2 b_i^{j-1}, & 2 \le j \le m, \\ u_1^j - \Delta x g^j, & i = 0,\ 1 \le j \le m, \\ u_{k-1}^j + \Delta x h^j, & i = k,\ 1 \le j \le m, \\ u_{0i}, & j = 0,\ 0 \le i \le k, \\ u_i^0 + \Delta t u_{1i}, & j = 1,\ 1 \le i \le k-1, \end{cases} \quad (11)$$

where $\pi = (\Delta t)^2/(\Delta x)^2$, and $\mu_i^j,\ \vartheta_i^j,\ \kappa_i^j,\ \lambda^j,\ \sigma^j$ are quadrature coefficients.

We introduce the adjoint variable v_i^j and define the auxiliary function

$$E = \bar{J} + \sum_{i=1}^{k}\sum_{j=2}^{m}\Big[2(1-\pi)u_i^{j-1} - u_i^{j-2} + \pi\left(u_{i+1}^{j-1} + u_{i-1}^{j-1}\right) + (\Delta t)^2 a_i^{j-1}u_i^{j-1} + (\Delta t)^2 b_i^{j-1}\Big]v_i^j + \sum_{j=1}^{m}\Big[(u_1^j - \Delta x g^j)v_0^j + (u_{k-1}^j + \Delta x h^j)v_k^j\Big] + \sum_{i=0}^{k}\Big[u_{0i}v_i^0\Big] + \sum_{i=1}^{k-1}[u_i^0 + \Delta t u_{1i}]v_i^1. \quad (12)$$

Using formula (11) from [27], we obtain the adjoint variables defined as follows:

$$v_i^j = \begin{cases} 2(1-\pi)v_i^{j+1} - v_i^{j+2} + \pi\left(v_{i-1}^{j+1} + v_{i+1}^{j+1}\right) & 2 \le i \le k-2, \\ +(\Delta t)^2 a_i^{j+1}v_i^{j+1} + \Delta t\Delta x\mu_i^j\left(u_i^{j+1} - z_i^{j+1}\right), & 1 \le j \le m-2, \\ 2(1-\pi)v_1^{j+1} - v_1^{j+2} + \pi v_2^{j+1} + v_0^j & i = 1, \\ +(\Delta t)^2 a_1^{j+1}v_1^{j+1} + \Delta t\Delta x\mu_i^j\left(u_1^{j+1} - z_1^{j+1}\right), & 1 \le j \le m-2, \\ 2(1-\pi)v_{k-1}^{j+1} - v_{k-1}^{j+2} + \pi v_{k-2}^{j+1} + v_k^j & i = k-1, \\ +(\Delta t)^2 a_{k-1}^{j+1}v_{k-1}^{j+1} + \Delta t\Delta x\mu_i^j\left(u_{k-1}^{j+1} - z_{k-1}^{j+1}\right), & 1 \le j \le m-2, \\ \pi v_1^{j+1} + \Delta t\Delta x\mu_1^j\left(u_1^{j+1} - z_1^{j+1}\right), & i = 0,\ 1 \le j \le m-2, \\ \pi v_{k-1}^{j+1} + \Delta t\Delta x\mu_{k-1}^j\left(u_{k-1}^{j+1} - z_{(k-1)}^{j+1}\right), & i = k,\ 1 \le j \le m-2, \\ \Delta x\Delta t\mu_i^j(u_i^j - z_i^j) + 2(1-\pi)v_i^m & 2 \le i \le k-2, \\ +\pi(v_{i-1}^m + v_{i+1}^m), & j = m-1, \\ \Delta x\Delta t\mu_i^j(u_i^j - z_i^j) + 2(1-\pi)v_1^m & i = 1, \\ +\pi v_2^m + v_0^j, & j = m-1, \\ \Delta x\Delta t\mu_i^j(u_i^j - z_i^j) + 2(1-\pi)v_i^m & i = k-1, \\ +\pi v_{i-1}^m + v_k^j, & j = m-1, \\ \Delta x\Delta t\mu_i^j(u_i^j - z_i^j) + \pi v_1^m, & i = 0,\ j = m-1, \\ \Delta x\Delta t\mu_i^j(u_i^j - z_i^j) + \pi v_{k-1}^m, & i = k,\ j = m-1, \\ \Delta x\Delta t\mu_i^j(u_i^j - z_i^j), & i = 0,\ j = m, \\ \Delta x\Delta t\mu_i^j(u_i^j - z_i^j) + v_0^m, & i = 1,\ j = m, \end{cases}$$

$$v_i^j = \begin{cases} \Delta x \Delta t \mu_i^j (u_i^j - z_i^j), & 2 \le i \le k-2,\ j=m, \\ \Delta x \Delta t \mu_i^j (u_i^j - z_i^j) + v_k^m, & i=k-1,\ j=m, \\ \Delta x \Delta t \mu_i^j (u_i^j - z_i^j), & i=k,\ j=m, \\ v_i^1 - v_i^2, & j=0,\ 1\le i\le k-1, \\ 0, & i=0,\ i=k,\ j=0. \end{cases}$$

Using formula (12) from [27], we get gradient coefficients

$$\frac{dE}{dh^j} = \begin{cases} \Delta t \rho \sigma^j h^j + \Delta x p_k^j, & 1\le j \le m, \\ 0, & j=0, \end{cases} \tag{13}$$

$$\frac{dE}{dg^j} = \begin{cases} \Delta t \gamma \lambda^j g^j - \Delta x p_0^j, & 1\le j \le m, \\ 0, & j=0. \end{cases} \tag{14}$$

$$\frac{dE}{da_i^j} = \begin{cases} \Delta x \Delta t \alpha \vartheta_i^j a_i^j + (\Delta t)^2 u_i^j v_i^{j+1}, & 1\le i\le k-1, \\ & 1\le j\le m-1, \\ 0, & i=0,\ i=k,\ \ 1\le j\le m-1, \\ 0, & j=0,\ j=m,\ \ 0\le i\le k, \end{cases} \tag{15}$$

$$\frac{dE}{db_i^j} = \begin{cases} \Delta x \Delta t \beta \kappa_i^j b_i^j + (\Delta t)^2 v_i^{j+1}, & 1\le i\le k-1, \\ & 1\le j\le m-1, \\ 0, & i=0,\ i=k,\ \ 1\le j\le m-1, \\ 0, & j=0,\ j=m,\ \ 0\le i\le k, \end{cases} \tag{16}$$

IV. GRADIENT BASED ALGORITHM

The gradient type algorithm is developed to obtain optimal control functions a^*, b^*, g^*, and h^* based on the conjugate gradient method [28] with obtained gradient coefficients. The algorithm to solve optimal control problem is outlined as follows:

Select arbitrary $a^{(0)}$, $b^{(0)}$, $g^{(0)}$, and $h^{(0)}$ and compute $u^{(0)}$, $v^{(0)}$. Compute

$$\left[\frac{dE}{da}\right]^{(0)},\ \left[\frac{dE}{db}\right]^{(0)},\ \left[\frac{dE}{dg}\right]^{(0)},\ \left[\frac{dE}{dh}\right]^{(0)},$$

and

$$S_a^{(0)} = -\left[\frac{dE}{da}\right]^{(0)},\ S_b^{(0)} = -\left[\frac{dE}{db}\right]^{(0)},$$

$$S_g^{(0)} = -\left[\frac{dE}{dg}\right]^{(0)},\ S_h^{(0)} = -\left[\frac{dE}{dh}\right]^{(0)}.$$

Step 1: Let $n=0$.

Step 2: Select $\varsigma_a^{(n)}$, $\varsigma_b^{(n)}$, $\varsigma_g^{(n)}$, and $\varsigma_h^{(n)} \ge 0$ such the

$$J(a^{(n)} + \varsigma_a^{(n)} S_a^{(n)}, b^{(n)} + \varsigma_b^{(n)} S_b^{(n)}, g^{(n)} + \varsigma_g^{(n)} S_g^{(n)}, h^{(n)} + \varsigma_h^{(n)} S_h^{(n)}) \le J(a^{(n)}, b^{(n)}, g^{(n)}, h^{(n)})$$

Step 3: Let

$$a^{(n+1)} = a^{(n)} + \varsigma_a^{(n)} S_a^{(n)},\ b^{(n+1)} = b^{(n)} + \varsigma_b^{(n)} S_b^{(n)},$$

$$g^{(n+1)} = g^{(n)} + \varsigma_g^{(n)} S_g^{(n)},\ h^{(n+1)} = h^{(n)} + \varsigma_h^{(n)} S_h^{(n)}.$$

Step 4: Compute $u^{(n+1)}$, $v^{(n+1)}$ and

$$\left[\frac{dE}{da}\right]^{(n+1)},\ \left[\frac{dE}{db}\right]^{(n+1)},\ \left[\frac{dE}{dg}\right]^{(n+1)},\ \left[\frac{dE}{dh}\right]^{(n+1)}.$$

Step 5: Compute the conjugate coefficient by

$$\chi_a^{(n)} = \frac{\int_Q \left(\left[\frac{dE}{da}\right]^{(n+1)}\right)^2 dQ}{\int_Q \left(\left[\frac{dE}{da}\right]^{(n)}\right)^2 dQ},\ \chi_b^{(n)} = \frac{\int_Q \left(\left[\frac{dE}{db}\right]^{(n+1)}\right)^2 dQ}{\int_Q \left(\left[\frac{dE}{db}\right]^{(n)}\right)^2 dQ}$$

$$\chi_g^{(n)} = \frac{\int_{\Omega_t} \left(\left[\frac{dE}{dg}\right]^{(n+1)}\right)^2 dt}{\int_{\Omega_t} \left(\left[\frac{dE}{dg}\right]^{(n)}\right)^2 dt},\ \chi_h^{(n)} = \frac{\int_{\Omega_t} \left(\left[\frac{dE}{dh}\right]^{(n+1)}\right)^2 dt}{\int_{\Omega_t} \left(\left[\frac{dE}{dh}\right]^{(n)}\right)^2 dt}.$$

Step 6: Calculate the direction of descent:

$$S_a^{(n+1)} = -\left[\frac{dE}{da}\right]^{(n+1)} + \chi_a^{(n)} S_a^{(n)},\ S_b^{(n+1)} = -\left[\frac{dE}{db}\right]^{(n+1)} + \chi_b^{(n)} S_b^{(n)},$$

$$S_g^{(n+1)} = -\left[\frac{dE}{dg}\right]^{(n+1)} + \chi_g^{(n)} S_g^{(n)},\ S_h^{(n+1)} = -\left[\frac{dE}{dh}\right]^{(n+1)} + \chi_h^{(n)} S_h^{(n)}.$$

Step 7: If $a^{(n+1)}$, $b^{(n+1)}$, $g^{(n+1)}$, and $h^{(n+1)}$ are optimal controls, then stop the process. Otherwise, take $n=n+1$ and go to step 2.

V. NUMERICAL SIMULATIONS

The gradient type algorithm is developed to obtain optimal control functions a^*, b^*, g^*, and h^* based on the conjugate gradient method [28] with obtained gradient coefficients. Let $\Omega_x = [0,1]$, $\Omega_t = [0,1]$ and $Q = \Omega_x \times \Omega_t$. Consider the wave equation:

$$\begin{aligned} &u_{tt} = u_{xx} + a(x,t)u + b(x,t),\ (x,t)\in Q, \\ &u_x(0,t) = g(t),\ u_x(1,t) = h(t),\ t\in\Omega_t, \\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x\in\Omega_x, \end{aligned} \tag{17}$$

where $u \in H_0^1(\Omega_x)$ is the disturbance of position x at time t, u_0, $u_1 \in L^2(\Omega_x)$ and a, b, g, h are control functions satisfying (2)-(5).

Let

$$a^{(0)}(x,t) = e^{xt},\ b^{(0)}(x,t) = \cos\left(\frac{\pi x}{2}\right)e^{3\pi t},\ g^{(0)} = t,\ h^{(0)} = 3t^2,$$

$u_0(x) = \cos^2(\pi x)$, $u_1(x) = 1$ and desired profile z is a solution of the following wave equation:

$$\begin{aligned} &u_{tt} = u_{xx},\ (x,t)\in Q, \\ &u_x(0,t) = 0,\ u_x(1,t) = 0,\ t\in\Omega_t, \\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x\in\Omega_x, \end{aligned} \tag{18}$$

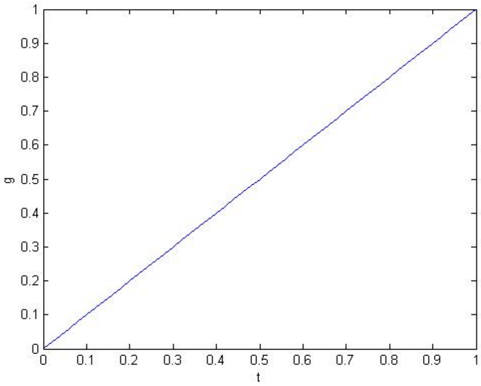

Fig. 1. Control g^0.

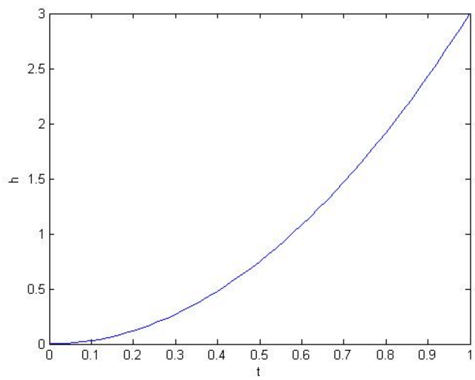

Fig. 2. Control h^0.

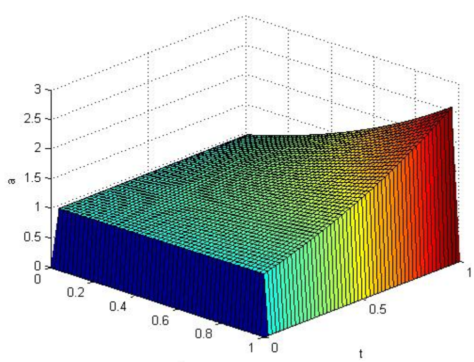

Fig. 3. Control a^0.

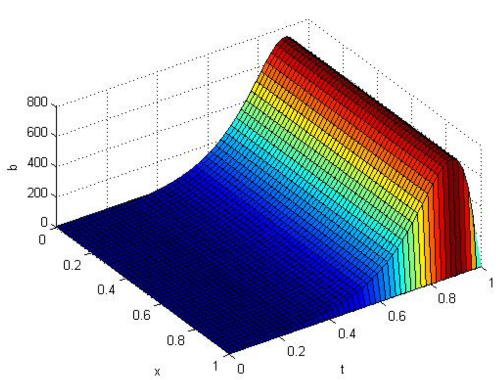

Fig. 4. Control b^0.

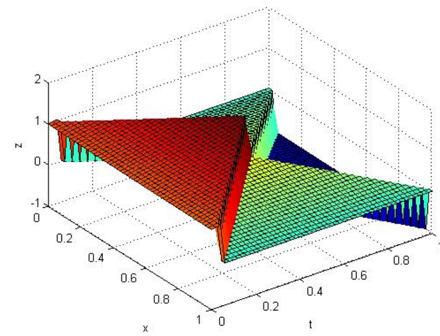

Fig. 5. Desired profile z.

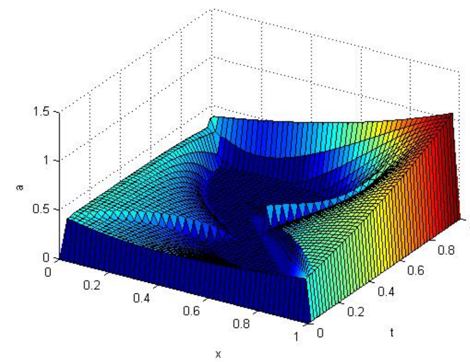

Fig. 6. Approximate Control a.

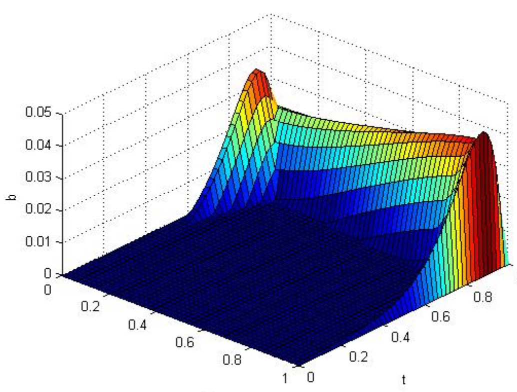

Fig. 7. Approximate Control b.

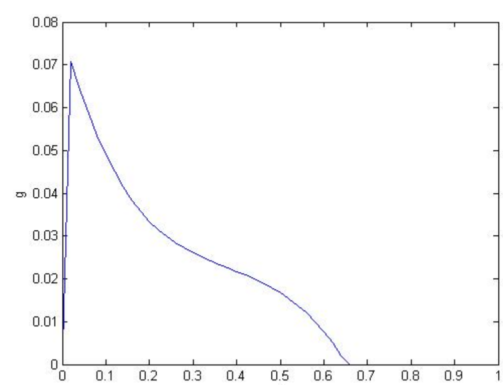

Fig. 8. Approximate Control g.

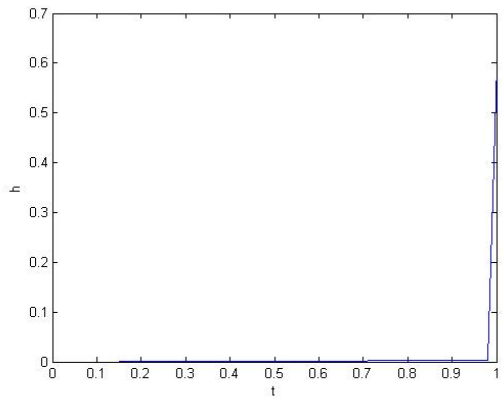

Fig. 9. Approximate Control h .

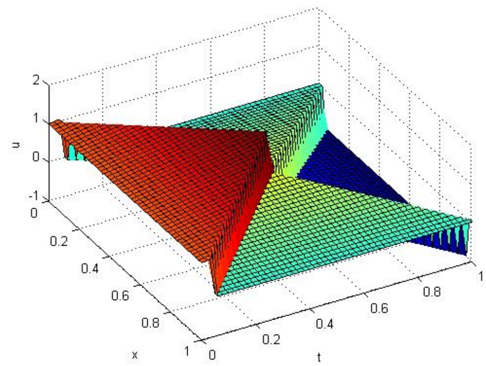

Fig. 10. Approximate controlled Solution u .

VI. CONCLUSIONS

In this paper, we studied a finite time optimal control problem involving hyperbolic system. Three types of controls were involved: multiplicative control a , interior control b , and boundary controls g , h . In the case of desired solution we have all control function at zero level. After using our algorithm, we have obtained approximate controls all of which are approaching zero level and approximate controlled solution is approaching the desired profile. This illustrates the effectiveness of our algorithm.

VII. REFERENCES

[1] V. Berdichersky, V. Jikov, and G. Papanicolcou, Eds., *Homogenization*, vol. 50 of Series on Advances in Mathematics for Applied Sciences, World Scientific Publishing, River Edge, NJ, USA, 1999.

[2] K. A. Lurie, *"Control in the coefficients of linear hyperbolic equations via spatio-temporal components,"* in Homogenization, vol. 50 of Series on Advances in Mathematics for Applied Sciences, World Scientific, River Edge, NJ, USA, 1999, pp. 285–315.

[3] K. A. Lurie, *"Some new advances in the theory of dynamic materials,"* Journal of Elasticity, vol. 72, no. 1–3, 2003, pp. 229–239.

[4] P. K. C. Wang, *"Feedback control of vibrations in an extendible wing,"* Dynamic Stability of Systems, Vol. 3 (1988), pp. 109-133.

[5] X. Pan and C. H. Hansen, *"Effect of end conditions on the active control of beam vibration,"* J. Sound Vib., Vol. 168 (1993), pp. 429-448.

[6] J. C. Bruch Jr., J. M. Sloss, S.A. Adali and I.S. Sadek, *"Orthotropic plates with shear rotation subject to optimal open-closed loop controls,"* J. Astronaut. Sci., Vol. 38 (1990), pp. 105-119.

[7] N. Basile and M. Mininni, *"An extension of the maximum principle for a class of optimal control problems in infinite-dimensional spaces,"* SIAM J. Control Optim., Vol. 28 (1990), pp. 1113-1135.

[8] H. O. Fattorini, *"Optimal control problems for distributed parameter systems in Banach spaces,"* Appl. Math. Optim., Vol. 28 (1993), pp. 225-257.

[9] J. C. Bruch Jr., S. Adali, J. M. Sloss and I. S. Sadek, *"Maximum principle for the optimal control of a hyperbolic equation in one space dimension,"* Part 2: application, J. Optim. Theory Appl., Vo. 87 (1995), pp. 287-300.

[10] J. C. Bruch Jr., I. S. Sadekc, S. Adali and J. M. Sloss, "*A maximum principle approach to optimal control for one-dimensional hyperbolic systems with several state variables,*" J. of the Frank. Inst., Vol. 343 (2006), pp. 480-494.

[11] M. Tatari and M. Dehghan, *"Identifying a control function in parabolic partial differential equations from over specified boundary data,"* Computers & Mathematics with Applications, vol. 53, no. 12, 2007, pp. 1933–1942.

[12] Y. Ou, T. C. Luce, E. Schuster, J. R. Ferron, M. L. Walker, C. Xu and D. A. Humphreys., "*Towards model-based current profile control at DIII-D*," Fusion Engineering and Design, Vol. 82 (2007), pp. 1153-1160.

[13] C. Xu, J. N. Dalessio, and E. Schuster, "*Optimal control of a parabolic PDE system arising in plasma transport via diffusivity-interior-boundary actuation*," 47th conference on decision and control, 2008.

[14] S. Lenhart, "*Optimal control of convective-diffusive uid problem*," Math. Models and Methods in Appl. Sci., Vol. 5 (1995), pp. 225-237.

[15] H. R. Joshi, *"Optimal control of the convective velocity coefficient in a parabolic problem,"* Nonlinear Analysis: Theory, Methods & Applications, vol. 63, no. 5–7, 2005, pp. 1383–1390.

[16] Y. Leredde, J.-M. Lellouche, J.-L Devenon and I. Dekeyser, "*On initial boundary conditions and viscosity coefficient control for Burgers equation*", Int. J. Num. Methods in Fluids, Vol. 28 (1998), pp. 113-128.

[17] H. Gao, "*Optimality condition of systems governed by semilinear parabolic equation*," Acta. Mathematica Sinica, Vol. 42 (1999), pp. 705-714.

[18] C. Xu and E. Schuster, "*Iterative design of suboptimal feedback control of bilinear parabolic PDE systems*," American control conference, 2009.

[19] C. Xu, Y. Ou, and E. Schuster, "*Sequential linear quadratic control of parabolic PDEs based on POD model reduction*," Automatica, Vol. 47 (2011), pp. 418-426.

[20] K. Ztot, E. H. Zerrik, and H. Bourray, "*Regional control problem for distributed bilinear systems: approach and simulations*," Int. J. Appl. Math. Comp. Sci., Vol. 21 (2011), pp. 499-508.

[21] S. Lenhart, M. Liang, and V. Protopopescu, *"Identification problem for a wave equation via optimal control,"* in Control of Distributed Parameter and Stochastic Systems, vol. 141 of IFIP Conference Proceedings, Kluwer Academic Publishers, Boston, Mass, USA, 1999, pp. 79–84.

[22] Lenhart, S. and Liang, M. *"Bilinear optimal control for a* wave equation *with viscous damping,"* Houston Journal of Mathematics, Vol. 26, no. 3, 2000, pp. 575–595.

[23] F. Maestre, A. Münch, and P. Pedregal, *"A spatio-temporal design problem for a damped wave equation,"* SIAM Journal on Applied Mathematics, vol. 68, no. 1, 2007, pp. 109–132.

[24] M. Liang, "*Bilinear optimal control for a wave equation,"* Mathematical Models & Methods in Applied Sciences, vol. 9, no. 1, 1999, pp. 45–68.

[25] A. Chambolle and F. Santosa, *"Control of the wave equation by time-dependent coefficient,"* ESAIM: Control, Optimization and Calculus of Variations, vol. 8, 2002, pp. 375–392.

[26] T. Abualrub, I. Sadek, and M. Abukhaled, *"Optimal Control Systems by Time-Dependent Coefficients Using CAS Wavelets,"* Journal of Applied Mathematics, Vol. 2009, Article ID 636271, 10pages.

[27] Y. Evtushenko, *"Computation of exact gradients in distributed dynamic systems,"* Int. J. Control, Vol. 6(1) (1997), pp. 81-89.

[28] L. S. Lasdon, S. K. Mitter, A. D. Waren, *"The Conjugate Gradient Method for Optimal Control Problems,"* IEEE Tran. On Automatic Control, Vol. AC-12, No. 2 (1967), pp. 132-138.

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Politics of Proxemics and Haptics in Work Space:
Handshakes and Power-play

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***Abstract* - Non -verbal communication has proved itself more powerful and credible than verbal communication. The behavior of non-verbal communication like Proxemics and Haptics were always studied in isolation. The paper is an attempt to study the interplay of these two tools in creating, maintaining and manipulating relationships, power index and dominance in work spaces.**

***Index Words*- Non-verbal, Proxemics, Haptics, Handshake**

When we 'celebrate' ourselves, as Walt Whitnman, had said As human beings, the first thing we celebrate is about the fact that we can think, we can ideate and we can communicate and thus can bond.

The importance of language is always emphasized upon. Diction, expressions, vocabulary, sentence syntax but still if we go through all the characteristic of Language, we will find that the Language has restriction. These restrictions limit the free flow of conversation words though powerful, precise and to the point are not enough. We need something more than that to express ourselves. We need a communication tool to reinforce the verbal tools, hence the importance and role of non-verbal Communication. it is this communication that goes beyond the words and creates a new world for the sender and the receiver both.

The "non-verbal communication involves those nonverbal stimuli, in a communication setting that are generated by both the source [speaker] and his or her use of the environment and that have potential message value for the source or receiver [listener] [1]

Mehrabian[2], however, proposes that about 93% of the information source may be generated from the non-verbal factors. And it is also true that we trust non-verbal communication more than verbal Communications. The cues given non-verbally are unconscious, involuntarily and thus more honest than the verbal communication which sometimes is voluntarily, conscious and thus manipulated.

Propagator of non-verbal communication Edward T. Hall opined " Have learned to depend more on what people do than what they say in response to a direct question, to pay close attention to that which cannot be consciously manipulated, and to look for patterns rather than content*."* [3]

Non-verbal communication performs six primary universal function viz,: Repeating, Complementing, Substituting, Reinforcing, Contradicting and Regulating. The different types of non-verbal communication tools includes: Kinesics, Haptics, Proxemics, Chronemics, Oculesics,Para-linguistic and artifacts .

This paper will dwell between the relationship between the Proxemics and Haptics and manipulation through the usage of the same

Proxemics is study of spaces. Spaces as non-verbal tool signify power, status, money, relationships. Under proxemics, areas or zones that human uses to protect themselves from intrusion and encroachment of outsiders were identified as :

Intimate Zone (0-18 feet) belongs to self, except close and loved ones , it is open to others when they are close and have warm relationship, who rush /

enter into the zones are actually intrusive whereas Personal Zone is reserved for the close friends and not intimate but other family members .Social Zone is meant for social interaction like that of acquaintance and other colleagues.

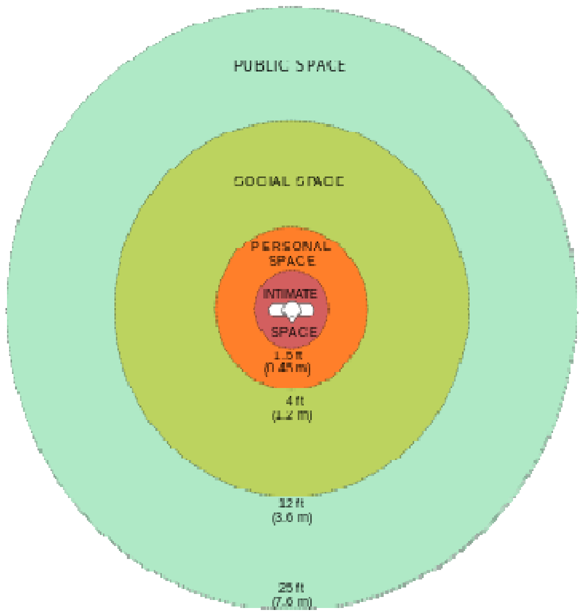

Fig.1: Diagram of Edward T. Hall's personal reaction bubbles (1966), showing radius in feet

These zones are the major key points in human behavior though they get changed according to the culture and relationship between the players. Hall noted, too, that different cultures do have their own rues of these zones.

In a work place where employee is spending almost 8-10 hours per day working- having tussles and conflicts .frequent ego rides and disruptive interpersonal relations keeping and still developing affinity and rapport becomes a challenge .

The term Haptics refers to --the sense of touch which come from the Greek ἅπτω which means "I fasten onto, I touch". It is a form of nonverbal communication. Humans love touches, touch signifies plethora of meaning, it is healing in nature and sometimes comforting. It can inflict trauma also and it can manipulate (negotiate) people and relationship.

By the clever use of proxemics and haptics sometimes unconsciously and sometimes consciously we assert dominance and negotiate effectively.

Most common form of haptics in an organization is Handshake. The simple act of holding another's hand with a grip is a conveys range of feeling starting from expressing congratulations, contractual agreement, farewell, greeting, dominance and power play.

Practical origin of the handshake comes from medieval Europe, where kings and knights would extend their hands to each other, and [grasp the] others hand as a demonstration that each did not possess concealed weapons and intended no harm to the other [1]. But now the more evolved from of handshake has changed this nonverbal tool as an essential and play-by-rule tool. It can make or break the deal, negotiation, impression and relations. It is a tool that is used to create the first impression and thus start an interaction. It is a welcoming sign of friendliness, hospitality, and trust [4].

There are different types of handshake, but this paper will only deliberate about the three types of two-hand handshakes that is

- Handshake touching wrist
- Handshake and Elbow grasp
- Handshake Shoulder holding

The Two-Handed Handshake was popularized by , John F. Kennedy, who discovered the two-handed handshake appears to be the warmest and build rapport easily.

Handshake touching wrist: Now let's assume a scenario where two co workers A and B are working ad they don't have very close relationship. A wants B to do some favour for him. To impart the feeling of closeness, proxemics and haptics is used very discreetly. When A person is shaking hand , to give little close feeling, he will touch the wrist of the B simultaneously.

Fig.2:http://www.indiabix.com/body-language/palm-gestures/

This action produces false sense of closeness as he is on the boundary of intimate zone of the other persona and the other hand is just inside the zone

Handshake and Elbow Grasp : The elbow grasp, is actually more manipulative and aggressive type of handshake. It transmits more feelings than the wrist hold.

The penetration of initiator's left hand encroaches not only the receiver's intimate zone but psychologically also convey to him that person is actually closed and hence friendly, intimate, and trustworthy .

Fig.3: US President Obama's with his predecessor, George W. Bush

The shoulder hold and the upper arm grip : This kind of handshake totally changes the equation between the "mere" acquaintance to good even great friends . a false impression created by snaking into and encroaching into receiver's close intimate zone . The handshake also changes into the submissive one and the overlapping of intimate spaces completes the act of manipulation.

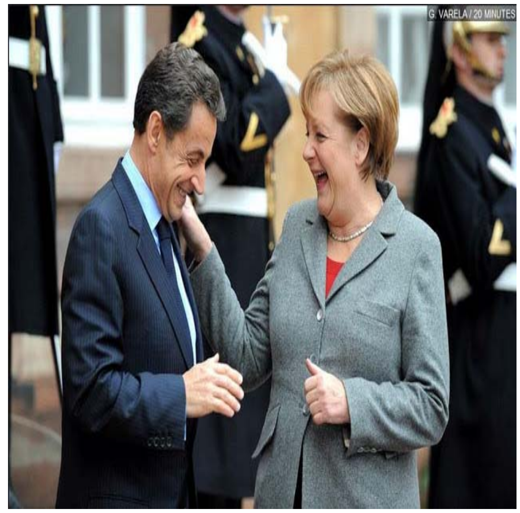

Fig.4 [Chancellor Angela Merkel of Germany and President Nicolas Sarkozy of France –HOLDING, FOR NOW Their alliance is trying to stabilize Europe but may become less essential in the near future

Source:NY times & http://bornatthecrestoftheempire.blogspot.in/2008/11/picture-of-day-2_11.html]

As said by Wallace Stegner , " It is touch that is the deadliest enemy of chastity, loyalty, monogamy, gentility with its codes and conventions and restraints. By touch we are betrayed and betray others ... an accidental brushing of shoulders or touching of hands ... hands laid on shoulders in a gesture of comfort that lies like a thief…"[5]

Thus touching also communicates control, power and status. People portrays high status with liberal and aggressive touching and invasion of intimate space in work place changes the relationships between the co- workers.

It is true that handshake give us the clue of person's motives, behavior, personality and impression, but it is also true that by cleverly manipulating this tool one can project false image and deviously manipulate the relationship in and across an organization.

REFERENCES

[1] Samovar, Larry A. & Porter, Richard E. 2000. Communication Between Cultures. Third edition. Foreign Language and Research Press, Brooks / Cole / Thomson Learning Asia

[2] Mehrabian, A. 1968. "Communication without Words" Psychology Today

[3] Hall, Edward T. (1966). The Hidden Dimension. Anchor Books. ISBN 0-385-08476-5.P.83

[4]Hall, Peter M. and Dee Ann Hall. "The Handshake as Interaction." SEMIOTICA v.45. Mouton Publishers: Amsterdam, 1983. pp. 249-264.

[5]Schriffin, Deborah. "Handwork as Ceemony." SEMIOTICA v.12. Mouton Publishers: Amsterdam, 1974. pp. 189-202.

[6] Wallace Stegnor, Angle of Reposr, Penguin Twentieth-Century Classics

http://soc302.tripod.com/soc_302rocks/id8.html (Chapter Five: Nonverbal Communication Web Poject-Kevin Andrew)

http://www.hum.utah.edu/communication/classes/fa02/1010-8/ch5.pdf

http://bornatthecrestoftheempire.blogspot.in/2008/11/picture-of-day-2_11.html

http://www.indiabix.com/body-language/palm-gestures/

http://www.nytimes.com/2011/01/16/magazine/16MerkelSarkozy-.html?pagewanted=all&_r=0

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Forestry Biomass Conservation with Synthetic Industry: A Mathematical Model

Manisha Chaudhary, Joydip Dhar

***Abstract*-- In this paper, a mathematical model is proposed and analyzed considering four compartment as pre-mature trees, nature trees, wood based industries and alternative synthetic industries. It is assumed that the pre-mature trees grow logistically, and it requires a fixed amount of time for maturity, finally the mature trees are used by the wood based industries. Further assumed that the synthetic industries grow at a constant rate and there is a competition between these two types of industries. The behaviour of the system near all feasible equilibrium are studied and it is observed that the maturation delay of trees plays an important role and the system exhibits a Hopf-bifurcation about the non-trivial equilibrium with respect to the delay parameter. Numerical justification of the analytical results is also shown in the last section.**

***Index Terms*-- Forest biomass, Wood based industries, Synthetic industries, Stability, Oscillating behaviour.**

I. INTRODUCTION

FORESTS plays an important role in every individuals life since origin, human population has depended upon forest for its various needs, be it food, fodder, fibre, fertilizers, medicine, construction material etc. This well known fact that the forest remainder left in forests after trees are rapidly cut down and bucked, such as branches, treetops, and thinned wood, are attracting attention as unused resources (approx. 3.7 million tons generated annually) for carbon neutral efforts. The usage of forest remainder biomass resources will greatly contribute to greenhouse gas mitigation and energy saving in the manufacturing industry, as well as to the maintenance of forests (i.e., new plantation of trees), leads to job creation and the vitalization of local economies, etc. Forest maintain the environment including atmospheric stability and in supplying the essential requirements of people all around the world. The main goal of forestry is to create and implement systems that allow forests to continue a sustainable provision of environmental supplies and services. The challenge of forestry is to create systems that are socially accepted while sustaining the resource and any other resources that might be affected. Also forests are well-known for the wealth of services they provide: supplying timber and non-timber forest products, mitigating climate change, preserving biological diversity, maintaining indigenous livelihoods, and providing for recreational and spiritual purposes. It is a well accepted fact that forestry resource plays a vital role in the development of any countries present and future. But it is being depleted by increased industrialization, over growth of population and associated pollution [1], [4]-[6], [9], [10], [16]. A typical example is the Doon Valley in the northern part of India where the forestry resources are being depleted by limestone quarries, wood and paper based industries, growth of human and livestock populations, expansion of forest land for agriculture and settlement, etc., threatening the ecological stability of the entire region [8], [4]. It has been noted that the forest bio-diversity loss and changes in climate are closely linked with deforestation [3], [17]-[19], [24]. When a resource-biomass is exposed to toxicant, the toxicant interferes with resource metabolism. Due to this, the intrinsic growth of the biomass may be adversely affected [21]-[23]. Deforestation and forest degradation, whether due to human activities or natural causes, result in carbon stock reductions and greenhouse gas emissions, as well as loss or impairment of other forest goods and services, threatening livelihoods, environmental functions and other socio-economic values. Forest conservation endeavours to rectify such deleterious impacts that cause forest degradation, and ultimately deforestation. Forest threats include fires, pests and diseases, poor management and harvesting, overexploitation, grazing and other disturbances [25]. There has been great interest during the last few decades in dynamical characteristics of population model and among these models, forest biomass system play an important role in population dynamics [15], [7], [8], [9], [10]. Many authors proposed various model considering populations that compete for common resources. [11]-[13]. Dhar et al. proposed various models to study the effect of industrialization on growth and existence of a biological species that depends partially or wholly on forestry resource in two homogeneous adjoining patches with or without migrations, act as a predator on the resource [7]- [9]. It has been noted here that to save forest based resources it is very important to search alternatives of the same. Keeping the aforementioned facts in mind, we propose and analyze a mathematical model based on forestry resource conservation with growth of synthetic industry. We consider an ecosystem where the wood based resource is being continuously depleted due to the industrialization.

To overcome from the world wide problem of deforestation, conservation of forestry resources, using synthetic is good alternative of wood based product as it is cheap, needs not much maintenance and the one most important thing that it looks fresher than a wood based product. Many researchers and experimentalists have concentrated on the stability of the forest biomass systems and more specifically they have investigated on the stability

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of such systems with time delays incorporated into the models [2], [20], [17]. Such delayed system has received great attention since time delay may have complicated impact on the dynamical behaviour of the system.

II. Mathematical Formulation

A four compartment mathematical model has been proposed to study the effect of maturation delay on forestry biomass, where β is maturation rate of premature trees $P(t)$, density of premature trees converts into mature trees $B(t)$. The state variables $W(t)$, $S(t)$ are densities of wood based industries and synthetic industries respectively. r is intrinsic growth rate
of premature trees, k is the carrying capacity of forest biomass, c_1, c_2 are competition effect of $W(t)$ and $S(t)$, α is depletion coefficient of forest biomass, α_1 is growth rate of wood based industries, Q is the constant rate of raw material supplied to synthetic industries, h_1 and h_2 are natural washout rate of wood based and synthetic industries respectively. Parameter descriptions are summarized in table 1 and the schematic flow diagram is shown in fig.1.

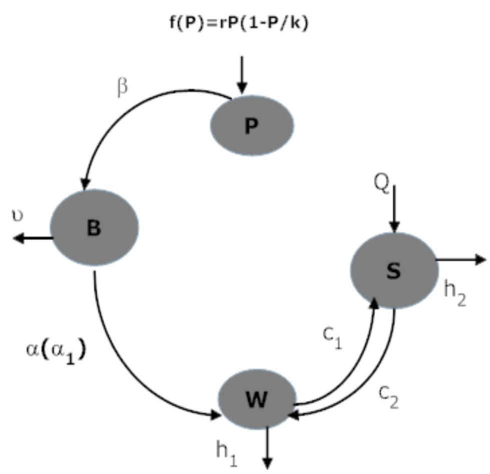

Fig.1: Schematic diagram

The model assumes that the synthetic industries are good alternative for the forest biomass. The model is being formulated with the help of following system of non-linear ordinary differential equations:

$$\frac{dP}{dt} = rP\left(1-\frac{P}{k}\right) - \beta P(t-\tau), \tag{1}$$

$$\frac{dB}{dt} = \beta P(t-\tau) - \alpha BW - \nu B, \tag{2}$$

$$\frac{dW}{dt} = \alpha_1 BW - c_1 WS - h_1 W, \tag{3}$$

$$\frac{dS}{dt} = Q - c_2 WS - h_2 S. \tag{4}$$

with non-negative initial conditions $P(0) > 0, B(0) > 0, W(0) > 0, S(0) > 0$. Here we will study two cases with and without maturation delay, i.e., $\tau = 0$ and $\tau \neq 0$ respectively.

In next sections, we will discuss possible steady state of the system, when $\tau = 0$.

TABLE I
PARAMETER AND VARIABLE DESCRIPTION

Parameter	Description
α	depletion rate of forest biomass due to the wood based industries
α_1	growth rate of wood based industries
β	maturation rate
c_1 and c_2	depletion rates of $W(t)$ and $S(t)$ due to competition
h_1	natural washout rate of wood based industries
h_2	natural washout rate of synthetic based industries
v	natural depletion rate of mature trees
k	carrying capacity of forest biomass
Q	constant rate of raw material available for synthetic industries
r	intrinsic afforestation rate
τ	maturation delay

III. Possible Steady State of the System

When $\tau = 0$, proposed system (1) to (4) has following biological feasible equilibria; namely,
(i) Axial equilibrium: $E_1 \equiv \left(0,0,0,\frac{Q}{h_2}\right)$; (ii) Positive Boundary equilibrium: $E2 \equiv \left(k - \frac{k\beta}{r}, \frac{(\beta k(r-\beta))}{rv}, 0, \frac{Q}{h_2}\right)$;
(iii) Positive interior equilibrium: $E^{*i} \equiv (P^*, B^*, W^*, S^*)$; $i = 1, 2$, where
$P^* = k\left(1-\frac{\beta}{r}\right)$,
$B^* = \frac{\beta k}{\alpha W^* - \nu}\left(1-\frac{\beta}{r}\right)$,
$S^* = \frac{Q}{c_2 W^* - h_2}$ and W^* is the roots of the quadratic equation:
$r\alpha c_2 h_1 W^{*2} + (\beta\alpha_1 k c_2 + r\alpha c_1 Q + \alpha h_1 r h_2 - \nu r h_1 c_2 -$ $r\alpha 1kc2W* + (\beta\alpha 1kh2 - r\alpha 1kh2 - \nu rh1h2) = 0.$ (5)
Equation (5) has unique (or two) positive root(s) if the coefficient of W∨ is negative and constant coefficient is negative (or positive). Finally, existence of E^{*i} ensured from

$$W^* > max\left\{\frac{\alpha}{\nu}, \frac{h_2}{c2}\right\}, r > \beta.$$

IV. Boundedness and Dynamic Behaviour without Delay

Now we will study the existence of the system and the boundedness of the solution. There are three biologically feasible equilibria for the system (1) to (4). Further we consider the function
$\omega(\tau) = P(t) + B(t) + W(t) + S(t)$ (6)
and substituting the values from (1) to (4), we obtain
$\frac{d\omega}{dt} = rP\left(1-\frac{P}{k}\right) - \alpha BW - \nu B - h_1 W + Q + \alpha_1 BW - c_1 WS - c_2 WS - h_2 S.$
If we choose a positive real number $\eta = \min(\nu, \gamma, h_1, h_2)$, then

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$$\frac{d\omega}{dt} + \eta W(\tau) \leq Q + rP\left(1 - \frac{P}{k} + \gamma P\right) = f(P).$$

Moreover, $f(P)$ has maximum at $P = k\left(\frac{r+\gamma}{2r}\right)$ and $f(P) \leq Q + k\left(\frac{r+\gamma}{2r}\right)\left(\frac{1}{2} + \frac{\eta}{2r}\right) = M$ (say), hence $\dot{w}(\tau) + \eta\omega(\tau) \leq M$. Now using comparision theorem and $\tau \to \infty$, we obtain $sup\ \omega(\tau) = \frac{M}{\eta}$. Therefore, the system is uniformly bounded in the region:

$$B = \left\{(P, B, W, S) \in R_4^+ :\ 0 \leq P(\tau) + \beta(\tau) + W(\tau) \leq \frac{M}{\eta}\right\}.$$

Variational matrix for the proposed model (1)-(4) corresponding to the equilibrium E_1 is given by

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta & -W\alpha - \nu & -B\alpha & 0 \\ 0 & W\alpha_1 & 0 & -c_1 W \\ 0 & 0 & -c_2 S & -h_2 - c_2 W \end{bmatrix}$$

And eigen-values for the equilibrium point E_1:

$$\lambda_1 = r - \beta, \lambda_2 = -\nu, \lambda_3 = 0, \lambda_4 = -h_2,$$

Eigen-values corresponding to the equilibrium point E_2 the general variational matrix for the proposed model (1)-(4) corresponding to the equilibrium point E^{*i} is

$$J_{E^{*i}} = \begin{bmatrix} r - \frac{2rP}{k} - \beta & 0 & 0 & 0 \\ \beta & -W\alpha - \nu & -B\alpha & 0 \\ 0 & W\alpha_1 & -h_1 - c_1 S + B\alpha_1 & -c_1 W \\ 0 & 0 & -c_2 S & -h_2 - c_2 W \end{bmatrix}$$

The characteristic equation corresponding to the Jacobean $J_{E^{*i}}$ will be of the form:

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \quad (7)$$

where $a_1 = h_1 + h_2 - \frac{2rP}{k} + c_1 S + c_2 W + W\alpha - \beta\alpha_1 + \beta$,

$a_2 = h_1h_2 - h_1 r - h_2 r + \frac{2h_1Pr}{k} + \frac{2h_2Pr}{k} + c_1h_2S + c_1rS + \frac{2c_1PrS}{k} + c_2h_1W - c_2rW + \frac{2c_2rPW}{k} + (h_1 + h_2 - r + c_1S +$ $\beta W\alpha + c2W2\alpha +$r – h2 – c2W1B$\alpha 1$+2$BPr\alpha 1$k+$2PrW\alpha$k+$(h1 + h2 + c1S + c_2W)\beta$

$a_3 = \frac{2h_1h_2Pr}{k} - h_1h_2r - c_1h_2rS + \frac{2h_2c_1PrS}{k} - c_2h_1rW + \frac{2c_2h_1PrW}{k} + (h_1h_2 - h_1r - h_2r)W\alpha + (h_1 + h_2)\frac{2PrW\alpha}{k} + (h_2 - r)c_1SW\alpha + \frac{2c_1PrSW\alpha}{k} + (h_1 - r)c2W^2\alpha + \frac{2c_2PrW2\alpha\alpha}{k} + Bh_2r\alpha_1 - \frac{2Bh_2Pr\alpha_1}{k} + Bc_1rW\alpha_1 - \frac{2Bc_2PrW\alpha_1}{k} + h_1h_2\beta + c_1h_2S\beta + c_2h_1W\beta + h_1W\beta + h_2W\beta\alpha + c_1SW\beta\alpha + c_2W2\beta\alpha - Bh_2 - Bc_2W\alpha_1\beta$,

$a4 = \frac{2c_2h_1PrW2\alpha}{k} + h_1h_2W\beta\alpha + c_1h_2SW\beta\alpha + c_2h_1W^2\beta\alpha - h_1h_2rW\alpha + \frac{2h_1h_2PrW\alpha}{k} - c_1h_2rSW\alpha + \frac{2c_1h_2PrSW\alpha}{k} - c_2h_1rW^2\alpha.$

Using Routh-Hurwitz criteria, the conditions $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, a_1a_2 - a_3 > 0$ and $a_1a_2a_3 - a_3^2 - a_1^2a_4 - a_1^2a_4 > 0$ and for the interior equilibrium point E^{*i} is locally asymptotically stable. To avoid complexity of analytical calculations, we perform it numerically using following set of parameters: $r = 1.5$, $k = 90$, $\beta = 0.7$, $c_1 = 0.4, c_2 = 0.32, h_1 = 0.24, h_2 = 0.2, \alpha_1 = 0.9, Q = 40, \alpha = 0.6, \nu = 0.15$. It is noticed that the interior equilibrium E_2 $(48, 5.8679, 9.29342, 12.6028)$exists and stable (see fig.2). As we decrease the growth rate parameter of wood industries i.e. $\alpha_1 = 0.41, r = 1.5, k = 90, \beta = 0.7, c_1 = 0.4, c_2 = 0.32, h_1 = 0.24, h_2 = 0.2, Q = 40$, $\alpha = 0.6$, $\nu = 0.15$ and the positive equilibrium point $E_2 = (48, 223.999, 0, 200)$ (see fig.3).

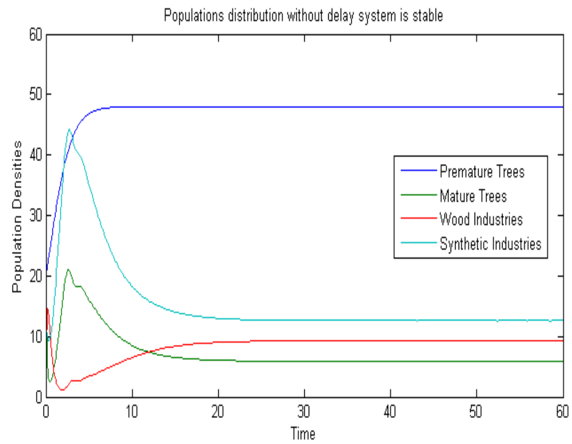

Fig. 2: Population model without delay system is stable.

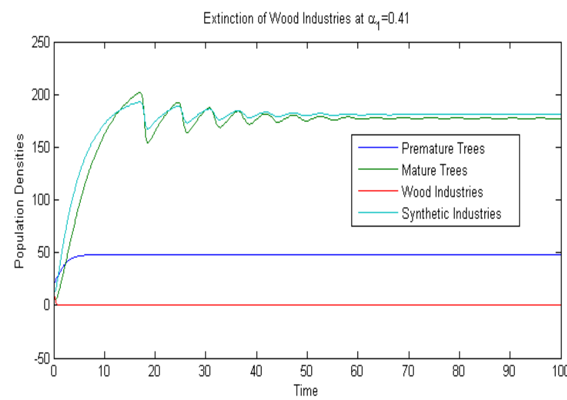

Fig. 3: Population distributions without delay and extinction of wood industries

V. Dynamic Behaviour with Delay

In this section, we will linearize our proposed system (1)-(4) around the positive (interior) equilibrium and investigate the stability of the equilibrium point. Due to biological interest of the system, we only consider the positive equilibrium. To study the local stability of the positive equilibrium $E_2 = (P_2^*, B_2^*, W_2^*, S_2^*) \to (P^*, B^*, W^*, S^*)$, we first use the linear transformation $P(t) = P^* + p(t)$, $B(t) = B^* + b(t)$, $W(t) = W^* + w(t)$, $S(t) = S^* + s(t)$where $p \ll 1, b \ll 1, w \ll 1$ and $s \ll 1$, for which the system turns out to be

$$\frac{dp}{dt} = rP(t)\left(1 - \frac{2P^*}{k}\right) - \beta p(t - \tau), \quad (8)$$

$$\frac{db}{dt} = \beta p(t - \tau) - \alpha B^* w(t) - \nu b(t), \quad (9)$$

$$\frac{dW}{dt} = \alpha_1 B^* w(t) + \alpha_1 W^* b(t) - c_1 W^* s(t) - h_1 w(t), \quad (10)$$

$$\frac{dS}{dt} = -c_2 W^* s(t) - c_2 S^* w(t) - h_2 s(t). \quad (11)$$

Variation matrix corresponding to the system of equations (8)-(11) is given as

$$J = \begin{bmatrix} r - \frac{2rP}{k} - \beta e^{-\lambda\tau} & 0 & 0 & 0 \\ \beta e^{-\lambda\tau} & -W\alpha - v & -B\alpha & 0 \\ 0 & W\alpha_1 & 0 & -c_1 W \\ 0 & 0 & -c_2 S & \frac{-Q}{S} \end{bmatrix}$$

whose characteristic equation is given by

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 + e^{-\lambda\tau}(b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4) = 0, \quad (12)$$

where $a_1 = -r + \frac{2rP^*}{k} + \frac{Q}{S^*} + W^*\alpha + v$,

$a_2 = \frac{2rP^*Q}{k} - \frac{Qr}{S^*} - c_1c_2S^*W^* - rW^*\alpha + \frac{2P^*W^*r\alpha}{k} + \frac{QW^*\alpha}{S^*} + B^*W^*\alpha\alpha_1 - rv + \frac{2P^*rv}{k} + \frac{Qv}{S^*}$,

$a3 = c_1c_2rS^*W^* - \frac{2c_1c_2P^*rS^*W^*}{k} - Qr\alpha S^*W^* + \frac{2P^*QrW^*\alpha}{kS^*} - c_1c_2S^*W^{*2}\alpha - B^*r\alpha\alpha_1W^* + \frac{2B^*P^*r\alpha\alpha_1W^*}{k} + B^*QW^*\alpha\alpha_1S^* - QrvS^* + \frac{2P^*Qrv}{kS^*} - c_1c_2S^*W^*v$, $a4 = c_1c_2rS^*W^{*2}v - \frac{2c_1c_2P^*rS^*W^{*2}\alpha}{k} - B^*QW^*r\alpha\alpha_1S^* + \frac{2B^*P^*r\alpha\alpha_1W^*}{kS^*} + c_1c_2rS^*W^{*2}v - \frac{2c_1c_2P^*rS^*W^{*2}v}{k}$,

$b_1 = \beta$, $b_2 = \left(\frac{Q}{S^*} + v + \alpha W^*\right)$, $b_3 = \beta\left(\frac{QW^*\alpha}{S^*} - c1c2S*W* + \alpha\alpha1B*W* + QvS*\right.$,

$$b_4 = \beta(-c_1c_2rS^*W^{*2}\alpha + B^*QW^*r\alpha\alpha_1S^* - c_1c_2rS^*W^*v).$$

Using Routh-Hurwitz criteria, conditions for the stationary point to be locally asymptotically stable are $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, and\ a_1 > 0, a_1a_2\,a_3 - a_3^2 - a_1^2a_4 > 0$ holds true.

Next we suppose that for $\tau = \tau^*$, the transcendental equation (12) has one purely imaginary root, $\lambda = i\omega$ with ω real and without loss of generality $\omega > 0$, then we have

$$i\omega^4 + a_1(i\omega)^3 + a_2(i\omega)^2 + a_3(i\omega) + a_4 + e^{-\lambda\tau}(b_1(i\omega)^3 + b_2(i\omega)^2 + b_3(i\omega) + b_4 = 0$$

i.e.

$$\omega^4 - ia_1\omega^3 - a_2\omega^2 + a_3 i\omega + a_4 + b_4\cos\omega\tau + ib_3\omega\cos\omega\tau - b_2\omega^2\cos\omega\tau - ib_1\omega^3\cos\omega\tau + ib_4\sin\omega\tau - b3\omega\sin\omega\tau - ib2\omega2\sin\omega\tau + b1\omega3\sin\omega\tau = 0,$$

Separating real and imaginary parts, we have

$$\omega^4 - a_2\omega^2 + a_4 + b_4\cos\omega\tau - b_2\omega^2\cos\omega\tau - b_3\omega\sin\omega\tau + b_1\omega^3\sin\omega\tau = 0,$$

$$-a_1\omega^3 + a_3\omega + b_3\omega\cos\omega\tau - b_1\omega^3\cos\omega\tau - b_4\sin\omega\tau + b_2\omega^2\sin\omega\tau = 0,$$

$$(\omega^4 - a_2\omega^2 + a_4) - (b_2\omega^2 - b_4)\cos\omega\tau - (b_1\omega^3 - b_3\omega)\sin\omega\tau = 0, \quad (13)$$

$$(a_3\omega - a_1\omega^3) - (b_1\omega^3 - b_3\omega)\cos\omega\tau + (b_2\omega^2 - b_4)\sin\omega\tau = 0, \quad (14)$$

taking,

$E = (\omega^4 - a_2\omega^2 + a_4)$,
$F = (b_2\omega^2 - b_4)$,
$G = (b_1\omega^3 - b_3\omega)$,
$H = (a_3\omega - a_1\omega^3)$.

We can write the above two equations (13) and (14) as

$$E - F\cos\omega\tau - G\sin\omega\tau = 0, \quad (15)$$

$$H - G\cos\omega\tau + F\sin\omega\tau = 0. \quad (16)$$

On squaring and adding equations (15) and (16) we have

$$E^2 + H^2 - F^2(cos^2\omega\tau + sin^2\omega\tau) - G^2(cos^2\omega\tau + sin^2\omega\tau) = 0$$

$$E^2 + H^2 - F^2 - G^2 = 0.$$

Doing some mathematical manipulation in equations (15) and (16) and on adding we get,

$$\cos\omega\tau = \frac{(EF+HG)}{F^2+G^2}.$$

From above equation we can easily find the critical value of $\tau = \tau_0$,

$$\tau_0 = \frac{1}{\omega_0}\left[Arc\cos\frac{(EF+HG)}{F^2+G^2}\right] \quad (17)$$

We compute the value of τ critical numerically using following set of parameters $r = 1.5, k = 90, \beta = 0.7, c_1 = 0.4, c_2 = 0.32, h_1 = 0.24, h_2 = 0.2, \alpha_1 = 0.9, Q = 40, \alpha = 0.6, v = 0.15$ by means of MATLAB software. Using (15) and (16) we compute $\omega = \omega_0 = 0.692$ and on substituting the value of ω_0 in (17) we obtained the threshold value $\tau = \tau_0 = 2.9$. The results are shown with different values of $\tau_0 = 1.8, 2.8, 3.75. 3.8$ in fig. 4.

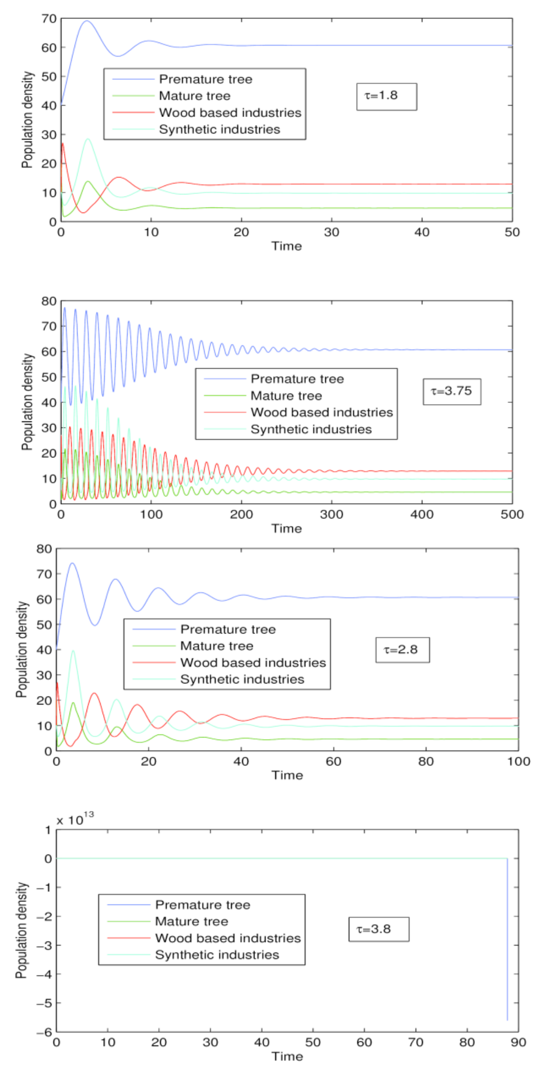

Fig.4: Population distribution with delay, for different values of τ=1.8, 2.8, 3.75, 3.8.

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VI. CONCLUSION

In this paper, we investigated a mathematical model of forestry biomass with a maturation delay and two types of industrializations. Dynamical behaviour of the system (1) - (4) with or without delay is obtained. By choosing the coefficient τ as a critical parameter and analyzing the associating characteristic equation. It is found that the critical parameter, maturation delay τ has a threshold value τ_0 for existence of the interior steady state. A numerical simulation is carried out for a particular set of parametric values. In fig. 2, we observed that the interior equilibrium point is stable when there is no delay. After introducing delay in the system, we get the threshold value $\tau_0 = 3.77$ and simulation is done for different values of $\tau = 1.8, 2.8, 3.75, 3.8$, which is shown in fig.4, it is observed that after crossing the threshold value $\tau_0 = 3.77$, system become unstable. Further comparing fig.2 and fig.4, we can easily visualized that in presence of maturation delay system requires more time to settle down to its equilibrium level and in presence of delay after a critical value of τ the interior equilibrium does not exist. We have also shown that as we decrease the growth rate parameter of wood based industries, the wood industries goes on extinction which is only possible by human awareness and try to search some wood alternatives.

VII. REFERENCES

[1] A. K. Misra, K. Lata, Modeling the effect of time delay on the conservation of forestry biomass, Chaos, Solitons & Fractals 46 (2013) 1–11.

[2] C. Celik, The stability and Hopf-Bifurcation for a predator-prey system with time delay. Chaos, Solution and Fractals, 37, (2008), pp. 87-99.

[3] D. (Ed) McHugh, Forests and climate change special report No. 6, World Wide Fund for Nature, WWF International, Gland Switzerland, (1991), pp. 16-18.

[4] J. B. Shukla, O.P. Misra, M. Agarwal,A. Shukla, Effect of pollution and industrial development on degradation of biomass resource: A mathemat-ical model with reference to Doon Valley. Mathl. Comput. Modell. Vol. 11, (1988), 910-913.

[5] J. B. Shukla, B. Dubey, Modelling the depletion and conservation of forestry resources: Effects of population and pollution. J. Math. Biol. 36, (1997), pp. 71-94.

[6] J. B. Shukla, K. Lata, A. K. Misra, Modelling the depletion of a renew-able resource by population and industrialisation: Effect of technology on its conservation, Natural Resource modelling, Vol. 24, (2011).

[7] J. Dhar, Population model with diffusion and supplementary forest resource in a two-patch habitat, Applied Mathematical Modelling 32 (7), (2008), pp. 1219-1235.

[8] J. Dhar, Modelling and analysis: the effect of industrialization on diffusive forestry resource biomass in closed habitat, African Diaspora J. Math, Vol.2(1), (2004), pp. 25-38.

[9] J. Dhar, H. Singh, Modelling the depletion of forestry resource by wholly dependent industrialization in two adjoining habitat, Kobe J. Mathematics, Japan, 21(2-1), (2004), pp. 1-13.

[10] J. Dhar, H. Singh, Diffusion of population under the influence industrialization in a twin-city environment, Mathematical Modelling and Analysis, 9(3), (2004), pp.124-133.

[11] J. Dhar and H. Singh, A two Species Competition Model with Diffusion and a Common Supplementary resource in two patch Habitat, Journal Advances in Theoretical and Applied Mathematics, Vol.2, Issue 2, 2007, pp. 103-116.

[12] J. Dhar, A prey-predator model with diffusion and a supplementary resource for the prey in a two-patch environment, Mathematical Modelling and Analysis,Vol.9, Issue 1, 2004, pp.9-24.

[13] J. Dhar and H. Singh, Modelling the conservation of forestry resource and animal population in two adjoining patches under the influence of deforestation, J. Ultra Scientist of Physical Sciences, Vol. 16, 3(M) , 2004, pp.253-260.

[14] J.D. Murray, Mathematical biology, Springer, Third Edition 2002.

[15] M. Agarwal, M. Fatima, H. I. Freedman, Depletion of forestry resources biomass due to industrilization pressure: A ratio dependent mathematical model, Journal of biological dynamics. Vol. 4, No. 4, (2010),pp. 381396.

[16] Manisha Chaudhary, Joydip Dhar, Govind Prasad Sahu, Mathematical Model of Depletion of Forestry Resource: Effect of Synthetic Based Industries, World Academy of Science Engineering and Technology Vol:7 No:4, (2013), pp.798-802.

[17] N. MacDonald, Time lags in biological models, Lecture notes in biomathematics 27 Heldelberg: Springer, (1978).

[18] O. Joshi, R. S. Mehmood, Factors affecting nonindustrial private forest landowners willingness to supply woody biomass for bioenergy, Biomass and Bioenergy 35, (2011), pp. 186-192.

[19] R.E. Munn,V. Fedorov, The Environmental Assessment, IISA Project Report, International Institute for Applied Systems Analysis, Laxenburg, Austria, Vol. I., (1986).

[20] S. Gakkhar, S. K. Sahani, A model for delayed effect of toxicanton resource biomass system. Chaos, Solutions and fractals, 36, (2007), pp. 912-922.

[21] S. Khare, O. P. Misra, J. Dhar, Role of toxin producing phytoplankton on a plankton ecosystem, Nonlinear Analysis: Hybrid Systems 4 (3), (2010), pp. 496-502.

[22] S. Sinha, O. P. Misra, J. Dhar, Modelling a predator-prey system with infected prey in polluted environment, Applied Mathematical Modelling 34 (7), (2010), pp. 1861-1872.

[23] Sudipa Sinha, O.P. Misra, Joydip Dhar, 2010, A two species completion model under the simultaneous effect of toxicant and disease, J. Nonlinear analysis: Real world applications (ELSEVIER), Vol.11(2), pp.1131-1142.

[24] S. L. Naeem, J. Thompson, S. P. Lawler, G. H. Lawton, R. M. Woodfin, Decline bio-diversity can alter the performance of ecosystem, Princeton University Press, New Jersey Nature 368, (1994), pp. 734-737.

[25] www.fao.org, Managing forest for climate change.

Modeling and Prediction of Computer and Video Game Playing Age Structured Population in USA

T K Sriram and Joydip Dhar

***Abstract*—Stage structured based population models have found many applications in the recent years. In this paper an age structured population model is developed and implemented to study the trends in the computer and video game playing population of US. The entire game paying population is divided into three compartments on the basis of the age group to which they belong. After simulating the mathematical model, a forecast of the number of game players in each stage as well as an approximation of the average age of game players in future has been made.**

***Keywords*—**Age structure, Forecasting, Mathematical modeling, Stage structure

I. Introduction

During a lifetime, an individual passes through a variety of stages. As he moves from one stage to another his choices and thoughts may keep on varying. Stage structure models are based on the assumption that individuals in a particular stage have similar characteristics. Depending on the factors such as age, size and awareness, many models based on stage structure have been proposed. S.Liu and L.Chen [1] argued that although stage structure models have received a lot of attention but models such as non autonomous stage structured models and those based on impulsive differential equations have not been extensively studied.

Models based on stage-structure can help in understanding and simulating natural processes. Rogers [2] argued that adoption of an innovation is a multi-step process and consists of five stages namely; awareness, interest, evaluation, trial and adoption. These five stages help a non-adopter in deciding whether to adopt an innovation or not, are facilitated through a number of communication channels. Researchers have proposed many stage structure models having immature and mature stages of particular population [3- 7]. In particular, W. G. Aiello et al. [5] has proposed another stage-structured model of population growth in which the time to maturity of the population was state dependent.

Age structured models may be used to understand and analyze the trends in a particular segment of population segregated on the basis of their ages. Norhayati and G.C. Wake [6] developed a non linear model by taking the effect of population density on age distribution of the population into consideration. Abia et al. [7] discussed some numerical approaches to solve problems on age-structured models. Motivated from few interesting compartmental models of population dynamics and epidemiology [8-10], in this paper, an age-structured compartmental model is developed by dividing the video and computer game players into three stages to analyze the trends in the three stages. Data required for estimating the parameters was collected from reports of Entertainment Software Association and United States Census Bureau [11, 12].

II. Mathematical Model

The entire video and computer game playing population of US was split up into three stages based on their age group. The first stage (N1) consisted of gamers between (2-17) years of age. The second stage (N2) consisted of gamers between (18-49) years of age and the third stage (N3) had gamers who are more than 49 years of age. We assume the existence of a frustration group which comprises of ex-gamers who may have got bored with the existing games and stopped playing. The rate at which gamers join this group from the three stages is taken as the frustration rate and the rate at which people in the frustration group rejoin the game playing population is taken as the rejoining rate. In this model, both the frustration and rejoining rates were considered to be fixed and independent of population size. We defined the rate of shifting of population from one stage to another as the movement shifting rate. Fig1 shows the schematic representation of the model.

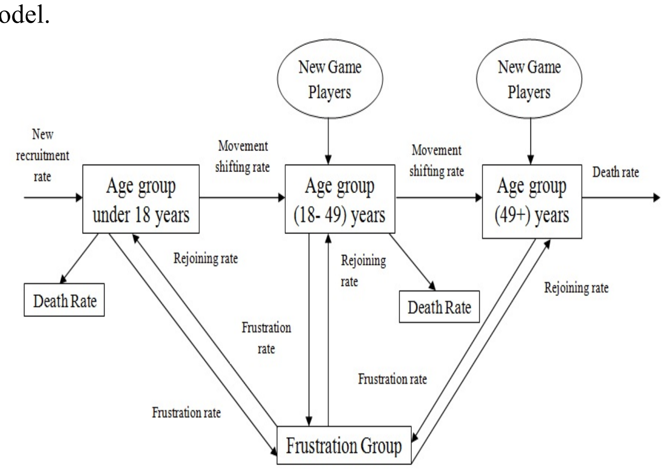

Fig. 1 Schematic model-Age-structured Game Playing Population

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Based on the above conditions, the following mathematical model is proposed:

$$\frac{dN1}{dt} = a - \alpha N1 - d1 - f1 + r1, \qquad (1)$$

$$\frac{dN2}{dt} = b + \alpha N1 - \beta N2 - d2 - f2 + r2, \qquad (2)$$

$$\frac{dN3}{dt} = c + \beta N2 - d3 - f3 + r3, \qquad (3)$$

$$\frac{dN4}{dt} = f1 + f2 + f3 - r1 - r2 - r3, \qquad (4)$$

where, N4 is population density of frustrated group, a, b and c are new recruitment rates in stage I, II, III respectively, again d1, d2 and d3 are death rates of gamers in Stage I, II, III respectively. Let as assume that α and β are shifting rates of gamers from Stage I to Stage II and Stage II to Stage III respectively. Finally, assume that f1, f2, f3 are frustration rates of gamers from Stage I, II, III and r1, r2, r3 are rejoining rate of non-gamers from frustration group into Stage I, II, III respectively.

Data regarding the video and computer game players was compiled from the "Essential Facts about the Computer and Video Game Industry" reports published by ESA during the years 2006-2010 and using US Census Bureau population data [11, 12]. Here, only the relevant data is presented.

Fig.2 shows the % of game playing population belonging to a particular age group. It can be seen that gamers in N2 stage form the largest group. Gamers in N1 and N3 stage contribute almost equally on % basis. Fig3 shows the graphical representation of the age-structured population densities in each stage. TABLE I provides the game playing population data for the years 2006-2010.

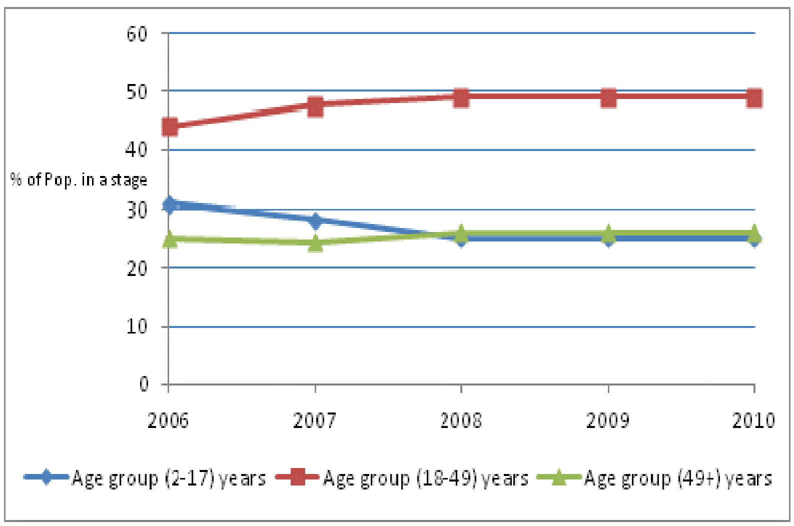

Fig. 2 Stage-wise % Game Playing population

TABLE I
Game Playing population

	2006	2007	2008	2009	2010
Total Game Playing Population	206029316	202058529	197843649	208764454	206859510
N1 - Gamers (2-17 years)	63869088	56980505	49460912	52191113	51714877
N2 - Gamers (18-49 years)	90652899	96179860	96943388	102294582	101361160
N3 - Gamers (49+ years)	51507329	48898164	51439348	54278758	53783472

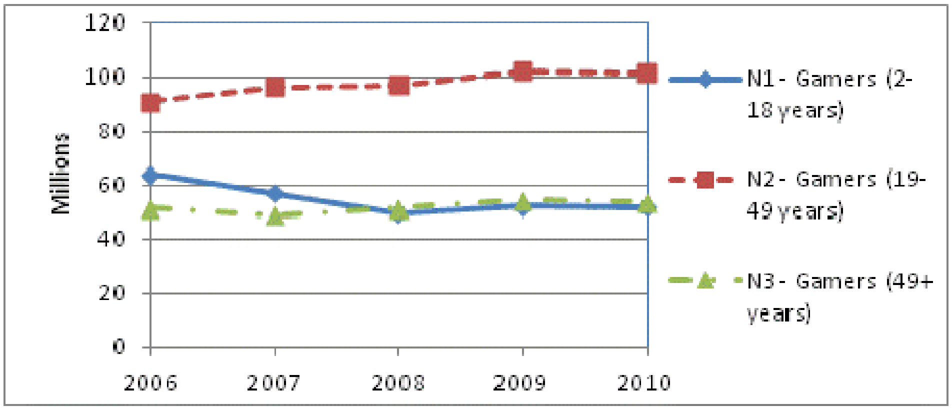

Fig.3 Age-structured Game Playing Population

III. Parameter Estimation of the Model

The movement shifting rate of gamers is assumed to be equal to the reciprocal of the number of years spent in a particular stage. Hence, α = 1/16 and β= 1/32. Also, as the frustration and rejoining rates were not separately available for the three stages, same frustration and rejoining rates (i.e., f = f1 = f2= f3 and r = r1 = r2= r3) were assumed. The model can now be represented as:

$$\frac{dN1}{dt} = a - \frac{N1}{16} - d1 - f + r, \qquad (5)$$

$$\frac{dN2}{dt} = b + \frac{N1}{16} - \frac{N2}{32} - d2 - f + r, \qquad (6)$$

$$\frac{dN3}{dt} = c + \frac{N2}{32} - d3 - f + r, \qquad (7)$$

$$\frac{dN4}{dt} = 3f - 3r . \qquad (8)$$

Three levels of frustration group were considered so as to understand how changes in the frustration group may affect the population in the three stages. Frustration group was assumed to consist of a certain percentage (8.33%, 10% and 15%) of the total game playing population. To obtain the L.H.S. of the above system of equations, curve fitting was done for a parabolic curve using the Least-Square method.

The equations of the curves are:

$$N1 = -1860655.8t^2 - 9837319.2t + 63869088 \qquad (9)$$

$$N2 = -511496.7t^2 + 5103347.8t + 90652899 \qquad (10)$$

$$N3 = 1604749.9t^2 - 3728702.6t + 51507329 \qquad (11)$$

$$N4(8.3\%) = 246060.6t^2 - 704940.7t + 17162242 \; (12)$$

$$N4(10\%) = 295391.1t^2 - 846267.9t + 20602931 \; (13)$$

$$N4(15\%) = 443086.4t^2 - 1269401.1t + 3090439 \; (14)$$

The difference in the actual and predicted values for N1 is shown in TableII. Since the % error values are within acceptable limits, the curve generated is appropriate.

TABLE II
Actual and Predicted Stage I Values

Year	Time(t)	Actual value	Predicted values y=pt^2+qt+r	Percentage Error
2006	0	63869088	63869088	0.00%
2007	1	56980505	55892424.6	1.91%
2008	2	49460912	51637072.8	-4.40%
2009	3	52191113	51103032.6	2.08%
2010	4	51714877	54290304	-4.98%

The values of dN/dt can be obtained by differentiating the above the equations with respect to t and putting t = 0 - 4 for the years 2006 - 2010 respectively. Table III represents the values for the three stages.

TABLE III
Year wise Rate of Change Values (N1, N2, N3)

Year	t	dN1/dt	dN2/dt	dN3/dt
2006	0	-9837319.2	5103347.8	-3728702.6
2007	1	-6116007.6	4080354.4	-519202.8
2008	2	-2394696	3057361	2690297
2009	3	1326615.6	2034367.6	5899796.8
2010	4	5047927.2	1011374.2	9109296.6

In order to simplify the model, we substitute a - d1 = K1, b - d2 = K2, c – d3 = K3 and f - r = K4 in the model equations (5-8). K1, K2 and K3 denote the year wise intrinsic growth rates in stages N1, N2 and N3 respectively and K4 is the intrinsic frustration rate. The model finally reduces to:

$$\frac{dN1}{dt} = K1 - \frac{N1}{16} - K4 \qquad (15)$$

$$\frac{dN2}{dt} = K2 + \frac{N1}{16} - \frac{N2}{32} - K4 \qquad (16)$$

$$\frac{dN3}{dt} = K3 + \frac{N2}{32} - K4 \qquad (17)$$

$$\frac{dN4}{dt} = 3K4 \qquad (18)$$

Ki’s values obtained by solving the above system for different frustration levels are provided in TABLE IV.

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TABLE IV
Year-wise K Values

	Year	2006	2007	2008	2009	2010
Frustration Group -8.3%	K4	-234980.22	-70939.82	93100.58	257140.99	421181.39
	K3	-6796585.92	-3595763.25	-246083.29	2960232.10	6362941.74
	K2	3709452.67	3453753.64	3088635.46	2226269.71	1367912.03
	K1	-6080481.42	-2625665.86	789711.58	4845701.15	8701288.40
Frustration Group -10%	K4	-282089	-85161.9	111765.5	308692.9	505620.3
	K3	-6843695	-3609985.3	-227418	3011784	6447380.65
	K2	3662344	3439531.56	3107300	2277821.6	1452350.94
	K1	-6127591	-2639887.9	808376.5	4897253.1	8785727.31
Frustration Group -15%	K4	-423134	-127742.77	167648.2	463039.1	758430.033
	K3	-6984739	-3652566.2	-171536	3166130.2	6700190.38
	K2	3521299	3396950.7	3163183	2432167.8	1705160.67
	K1	-6268635	-2682468.8	864259.2	5051599.3	9038537.05

It is observed that larger the level of the frustration group, higher is the intrinsic frustration rate (K4). On taking a larger frustration group, higher intrinsic growth rate values (K1, K2 and K3) are obtained. Initial negative values of K4 indicate that during 2006-2007 more people were rejoining the game playing population. However, from 2008 onwards, the number of people joining the frustration group outnumbered the people rejoining the game playing population. It is evident from TABLE IV that the intrinsic growth rates in Stages 1 and 3 were continuously increasing and changed from negative to positive over the years. Also, the intrinsic growth in Stage 2 showed a downward trend but remained positive.

Assuming the frustration group comprises of 8.33% of the game playing population for a given year, the model was simulated for the period (2006-2010) by substituting the Ki values in the model equations. Since the simulated trend lines as shown in Fig4 are very much similar to the actual trend lines in Fig3, this model is able to represent the trends in the game playing population correctly.

The model was further simulated till the year 2016 (Fig 5). The results show that there would be a growth in the number of gamers in all the three stages with N3 stage witnessing the most significant increment in game players. Its population would become nearly equal to that of N2 by 2016.

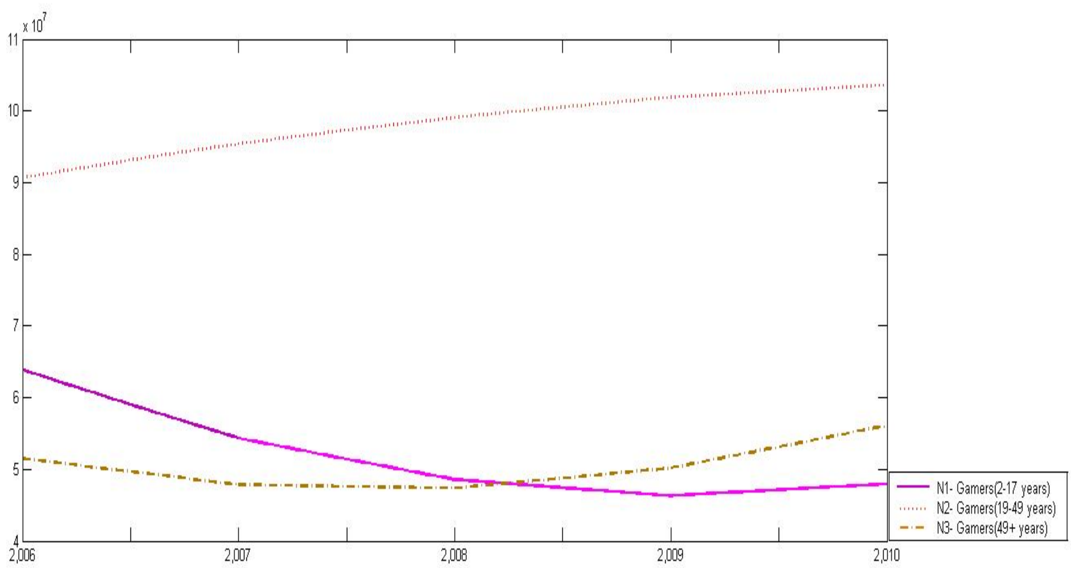

Fig. 4 Model Simulation 2006-2010

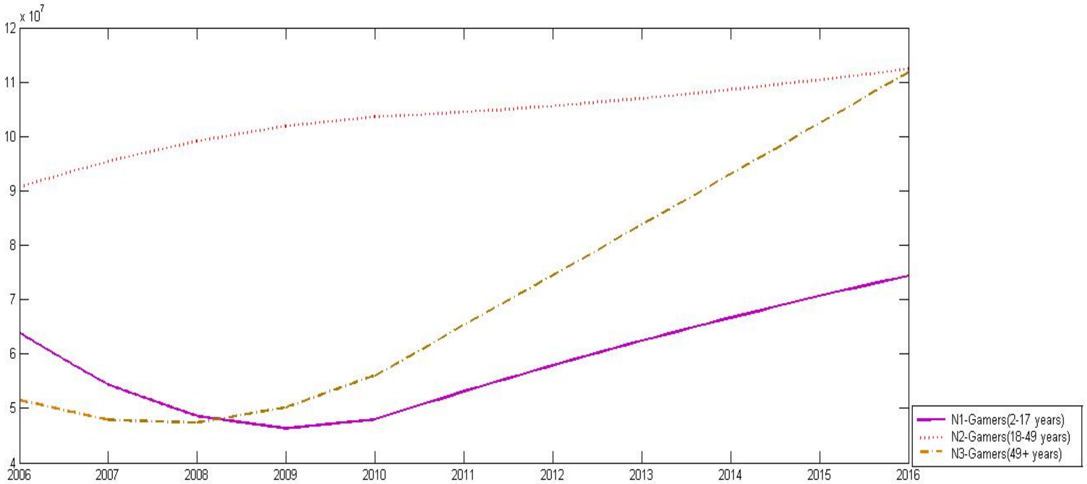

Fig. 5 Model Forecast 2006-2016

The forecasted population densities of the three stages are shown in TABLE III.

TABLE V
PREDICTION OF GAME PLAYING POPULATION

Year	N1(10^8)	N2(10^8)	N3(10^8)
2011	0.5306	1.0445	0.6528
2012	0.5787	1.0559	0.7450
2013	0.6239	1.0697	0.8376
2014	0.6664	1.0859	0.9307
2015	0.7063	1.1040	1.0243
2016	0.7438	1.1240	1.1186

This model was then used to calculate the average age of game players by 2016. Since, the population was assumed to be evenly distributed across all ages within a particular stage, the average of the game players can be calculated by taking the weighted average of the median age of gamers in the three stages. It was found that by 2016, the average of gamers would be around 39.5 years.

IV. Conclusions

As per the simulation results of the proposed compartmental mathematical, we found that in the coming years there will be a rise in the number of gamers attributed mainly to the N2 and N3 stages. By 2016, gamers in N1 stage will form only around 25% of the total gamers and the average age of gamers would become around 39.5 years. Hence, we may see a more games being developed to attract the upper age group stages. Also, efforts need to be put in understand the reasons behind the increase in frustration rate among the gamers to prevent the existing gamers from moving into the frustration group.

In future, this model can be extended to analyze the trends for a particular class of games. This model can be simulated by considering the exact frustration and rejoining rates for the three stages as well as with exact movement shifting rates subject to availability of data. Also, here we considered all the stages in isolation to each other and to the external environment. The model can be extended by incorporating the impact of external environment on a particular stage, for instance, the financial condition of the people in respective stages.

References

[1] S. Liu and L. Chen, "Recent Progress on Stage-Structured Population Dynamics," Mathematical and Computer Modelling, vol. 36, no. 11, pp. 1319-1360, 2002.

[2] E. Rogers, Diffusion of innovations, New York: 4th Edition, The Free Press, 1995.

[3] W. G. Aiello and H. Freedman, "A time-delay model of single-species growth with stage structure," Mathematical Biosciences, vol. 101, no. 2, p. 139–153, 1990.

[4] J. Dhar, K. S. Jatav, "Mathematical analysis of a delayed stage-structured predator–prey model with impulsive diffusion between two predators territories", Ecological Complexity (Elsevier) (in press), 2012.

[5] W. G. Aiello, H. I. Freedman and J. Wu, "Analysis of a Model Representing Stage-Structured Population Growth with State-Dependent Time Delay," SIAM Journal on Applied Mathematics, vol. 52, no. 3, pp. 855-869, 1992.

[6] Norhayati and G. C. Wake, "The solution and the stability of a nonlinear age-structured population model," The ANZIAM Journal, vol. 45, no. 02, pp. 153-165, 2003.

[7] L. Abia, O. Angulo and J. López-Marcos, "Age-structured population models and their numerical solution," Ecological Modelling, vol. 188, no. 1, p. 112–136, 2005.

[8] J. Dhar and A.K. Sharma, "The role of viral infection in phytoplankton dynamics with the inclusion of incubation class", Nonlinear Analysis: Hybrid Systems, vol. 4, no. 1, pp. 9-15, 2010.

[9] J. Dhar, "A prey-predator model with diffusion and a supplementary resource for the prey in a two-patch environment", Mathematical Modelling and Analysis, vol. 9, no. 1, pp. 9-24, 2004.

[10] S. Sinha, O.P. Misra and J. Dhar, "Modelling a predator–prey system with infected prey in polluted environment", Applied Mathematical Modelling, vo. 34, no.7, pp. 1861-1872, 2010.

[11] ESA, "Essential Facts about the Computer and Video Game Industry," Entertainment Software Association (ESA), USA, 2006-2010.

[12] D. U.S., "http://www.census.gov/2010census/," 28 11 2012. [Online]. Available: http://www.census.gov/2010census/.