Stress Distribution around Oval Shaped Hole in Infinite Composite Plate

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Abstract

In this paper, the solution of stress distribution around oval shaped hole is obtained using Muskhelishvili's complex variable approach. The oval shape is mapped outside unit circle using conformal mapping. The stress functions are obtained by evaluating Schwartz integral. The results are compared with literature and they are in good agreement. The various parameters like effect of stacking sequence, material, and loading angle are studied for symmetrical and unsymmetrical laminates for in-plane loading.

1. Introduction

Composite materials found wide applications in many areas of engineering due to advantage of strength to weight ratio, directional stiffness, wear resistance and corrosion resistance. The solution of the problem of stress analysis for composite material is also fascinating to researcher for many years. The solution for stress analysis becomes difficult when any hole or discontinuity is taken into account along with different boundary conditions. The discontinuities can arise due to environmental effect like corrosion or wear. In many applications, holes are created because of design requirement like window or opening to provide access into internal of the components. Such discontinuity will produce very high stresses and stress concentration when it is subjected to loading and may result into catastrophic failure if such stresses goes beyond strength value. Hence, it is necessary to establish methodology to solve problem of stress analysis around holes considering shape of the hole, type of loading and material properties.

The problem of stress analysis around circular hole for isotropic material was first solved by Kirsch [1] in 1898 by taking Airy's stress functions. Muskhelishvili [2] introduced complex variable method which is basis for many solutions in stress analysis. Savin [3] has considered Muskhelishvili's [2] approach and provided solution for isotropic and anisotropic media for both in-plane loading and moment around holes. Savin [3] presented solution for triangular, rectangular, square and many other shapes with general mapping function. Lekhnitskii [4] presented formulations for both in-plane loading and moment for anisotropic plate. The series approach is taken in the solution and hole shapes like circular, elliptical, rectangular, oval are solved for different anisotropic materials. Petrou and Theocaris [5] have considered effect of equilateral triangular hole in an infinite isotropic media and used Schwartz-Christoffal transformation for mapping of hole. Simha and Mohapatra [6] used conformal mapping to solve for stress concentration for polygonal hole.

Daoust and Hoa [7] and Ukadgaonker and Rao [8] presented the general solution for triangular hole in an anisotropic plate. Ukadgaonker and Awasare [9] presented novel method of two stage stress function by adopting Muskhelishvili's [2] complex variable method. Ukadgaonker and Rao [8], [10] provided solution for triangular and other hole shape with help of general mapping function and used biaxial loading condition given by Gao [12] to consider loading at infinity. The constants in mapping function decide shape and size of the hole. Stress functions and boundary conditions are obtained in general form by Schwarz Integral and solutions are given for different arrangement of laminate. General solutions for square and rectangular cutouts are obtained by Rao et al. [13] for anisotropic plates. The solution is given for isotropic as well as symmetric laminates along with failure strength using Tsai – Hill, Hasin-Rotem and Tsai – Wu criteria. Rezaeepazhand and Jafari [14] studied the effects of cutout on the load bearing capacity and stress concentration on plates. The stress analysis is done with different cutout. The results based on analytical solution are compared with the results obtained using finite element methods. Batista [15] had adopted Muskhelishvili's [2] formula to include complexity in the shape of cutout. Many examples of complex shapes are shown in this work and Schwartz-Chistoffel mapping is used for hole shape. Sharma [16] has considered cases of circular, elliptical and triangular hole for anisotropic plate. Effect of fiber angle, loading angle, material parameter and number of terms is studied in this work.

In present method, the solution for the stress distribution around oval shaped hole in anisotropic plate for in-plane loading is presented. Effect of various parameters like material properties, ply angle, stacking sequence, type of loading etc are presented for various composite materials.

2. Analytical formulation

The outside area of oval shaped hole in z-plane is mapped on to the exterior of unit circle in ζ-plane by using the mapping function in equation (1) as below:

$$w(\xi) = \frac{R}{2}\left[(1+c)\xi + \frac{(1-c)}{\xi} + \frac{2\varepsilon}{\xi^3}\right], \tag{1}$$

where $\xi = e^{i\theta}$, *c* is positive and less than unity and ε is negative and less than unity. Set of combinations of *c* and ε are derived to obtain the oval shape of different size.

A thin anisotropic plate is considered under generalized plane stress condition. The mean values of strains along thickness of plate can be represented by generalized Hooke's law as per equation (2).

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{2}$$

Representing σ_x, σ_y and τ_{xy} as Airy's stress function *U(x, y)* and using compatibility condition, following characteristic equation is obtained.

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0 \tag{3}$$

The roots of the above equation are

$$\begin{aligned} s_1 &= \alpha_1 + i\beta_1, \\ s_2 &= \alpha_2 + i\beta_2, \\ s_3 &= \alpha_3 + i\beta_3, \\ s_4 &= \alpha_4 + i\beta_4. \end{aligned} \tag{4}$$

The value of complex parameter s_1, s_2, s_3 and s_4 depend on the coefficients a_{ij}.*i,j=1, 2, 6* of the anisotropic plate.

The stress function *U(x, y)* is given as:

$$U(x, y) = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)}. \tag{5}$$

Taking stress functions, $\phi(z_1), \psi(z_2)$ and their conjugates $\overline{\phi(z_1)}, \overline{\psi(z_2)}$ as below,

$$\frac{dF_1}{dz_1} = \phi(z_1), \frac{dF_2}{dz_2} = \psi(z_2),$$
$$\frac{d\overline{F_1}}{d\overline{z_1}} = \overline{\phi(z_1)}, \frac{d\overline{F_2}}{d\overline{z_2}} = \overline{\psi(z_2)}, \tag{6}$$

and solving into Airy's stress function, σ_x, σ_y and τ_{xy} is obtained as below:

$$\sigma_x = 2\,\mathrm{Re}\left[s_1{}^2\phi'(z_1) + s_2{}^2\psi'(z_2)\right],$$
$$\sigma_y = 2\,\mathrm{Re}\left[\phi'(z_1) + \psi'(z_2)\right], \tag{7}$$
$$\tau_{xy} = -2\,\mathrm{Re}\left[s_1\phi'(z_1) + s_2\psi'(z_2)\right].$$

3. Solutions for Stress Function

The anisotropic plate containing oval shape hole is subjected to $\sigma_x^\infty = \lambda\sigma, \sigma_y^\infty = \sigma, \tau_{xy}^\infty = 0$ at infinity. The solution is divided into two stages to get stress functions.

In first stage solution, the plate is assumed to be hole free and load at infinity is considered. The first stage solution can be given by following equation:

$$\phi_1(z_1) = B^* z_1,$$
$$\psi_1(z_2) = (B'^* + iC'^*)z_2. \tag{8}$$

The constant B^*, B'^* and C'^* can be obtained by substituting equation (8) into equation (7). The boundary condition around imagery hole in the plate is given as below:

$$f_1 = 2\,\mathrm{Re}\left[\phi_1(z_1) + \psi_1(z_2)\right], \tag{9}$$
$$f_2 = 2\,\mathrm{Re}\left[s_1\phi_1(z_1) + s_2\psi_1(z_2)\right].$$

Since the hole boundary is traction free, the negative boundary conditions $-f_1$, $-f_2$ is considered in the absence of remote loading to obtain second stage stress functions $\phi_0(z_1), \psi_0(z_2)$. The solution of $\phi_0(z_1), \psi_0(z_2)$ is obtained by Schwarz formula as below:

$$\phi_0(\xi) = \left[\frac{a_3(1+c)}{\xi} + b_3\left(\frac{1-c}{\xi} + \frac{2\varepsilon}{\xi^3}\right)\right],$$
$$\psi_0(\xi) = -\left[\frac{a_4(1+c)}{\xi} + b_4\left(\frac{1-c}{\xi} + \frac{2\varepsilon}{\xi^3}\right)\right], \tag{10}$$

where,

$$a_3 = \frac{1}{(s_1 - s_2)}\left[s_2(K_1 + \overline{K_2}) - (K_3 + \overline{K_4})\right],$$
$$b_3 = \frac{1}{(s_1 - s_2)}\left[s_2(K_2 + \overline{K_1}) - (K_4 + \overline{K_3})\right],$$
$$a_4 = \frac{1}{(s_1 - s_2)}\left[s_1(K_1 + \overline{K_2}) - (K_3 + \overline{K_4})\right],$$
$$b_4 = \frac{1}{(s_1 - s_2)}\left[s_1(K_2 + \overline{K_1}) - (K_4 + \overline{K_3})\right], \tag{11}$$

$$K_1 = \frac{R}{2}\left[B^* a_1 + (B'^* + iC'^*)a_2\right],$$
$$K_2 = \frac{R}{2}\left[B^* b_1 + (B'^* + iC'^*)b_2\right],$$
$$K_3 = \frac{R}{2}\left[s_1 B^* a_1 + s_2(B'^* + iC'^*)a_2\right],$$
$$K_4 = \frac{R}{2}\left[s_1 B^* b_1 + s_2(B'^* + iC'^*)b_2\right]. \tag{12}$$

The final solution for stress functions $\phi(z_1), \psi(z_2)$ is given by super position of first and second stage stress function as below.

$$\phi(z_1) = \phi_1(z_1) + \phi_0(z_1),$$
$$\psi(z_2) = \psi_1(z_2) + \psi_0(z_2). \tag{13}$$

The stresses in Cartesian coordinates given by equation (13) are converted into polar coordinate by using transformation.

4. Results and discussion

The solution given by equation (13) is general solution for oval shape hole and by taking different value of c and ε, size of hole can be changed. The numerical results are obtained for Graphite/epoxy (E_1=181GPa, E_2=10.3GPa, G_{12}=7.17GPa and v_{12}=0.28) and Glass/epoxy (E_1=47.4GPa, E_2=16.2GPa, G_{12}=7.0GPa and v_{12}=0.26). Some of the results are obtained for isotropic plate (E=207GPa, G=79.3GPa and v=0.3) for comparison.

The size of the oval is decided by factor c and ε and its outside is mapped into unit circle by equation (1). The stiffness matrix for given composite is obtained by equation (2) and constant of anisotropy is obtained using equation (3). The values of B^*, B'^* and C'^* is obtained and first and second stage stress functions are calculated using equation (8) and (10) respectively. Table 1 provides the value of constants of anisotropy for these materials.

Fiber angle	Graphite/epoxy		Glass/epoxy	
	s_1	s_2	s_1	s_2
0	-0.0000 + 4.8936i	0 + 0.8566i	0.0000 + 2.3960i	-0.0000 + 0.7139i
90	-0.0000 + 1.1674i	0.0000 + 0.2043i	-0.0000 + 1.4007i	0.0000 + 0.4174i
0/90	-0.0000 + 3.6403i	0.0000 + 0.2747i	-0.0000 + 2.0142i	0.0000 + 0.4965i
45/-45	-0.8597 + 0.5109i	0.8597 + 0.5109i	-0.6045 + 0.7966i	0.6045 + 0.7966i
$0_4/90_4$)s	0.9231 + 1.1809i	-0.9231 + 1.1809i	-0.5390 + 1.0865i	0.5390 + 1.0865i

(0/90/45/-45)	-0.0000 + 1.0000i	0.7271 + 1.3876i	0.0000 + 1.0000i	0.2603 + 1.2054i

Table 1. Constants of Anisotropy

To validate the present method, the results for isotropic and plywood single layer plate is considered. The load is applied along small axis of oval opening. The results are compared with Lekhnitskii [4] solution in Table 2 and Figure 1 shows the distribution of stresses around hole for plywood plate. The results are in good agreement with literature.

	Isotropic Material		Ply wood	
	Present method	**Lekhnitskii**	**Present method**	**Lekhnitskii**
Maximum	4.42	4.44	6.76	6.39
Minimum	-1.02	-0.90	-1.41	-1.20

Table 2. Comparison of stress concentration values

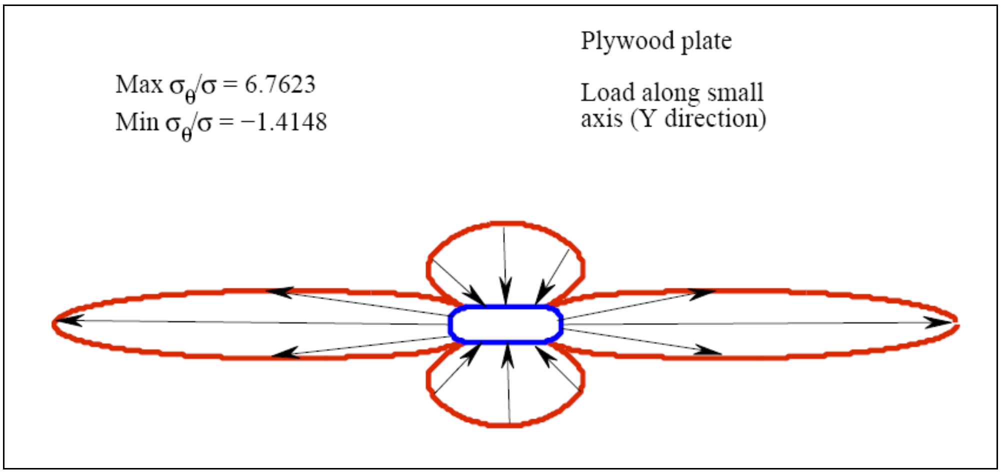

Figure 1. Stress distribution around oval for Plywood plate

4.1. Effect of Stacking Sequence

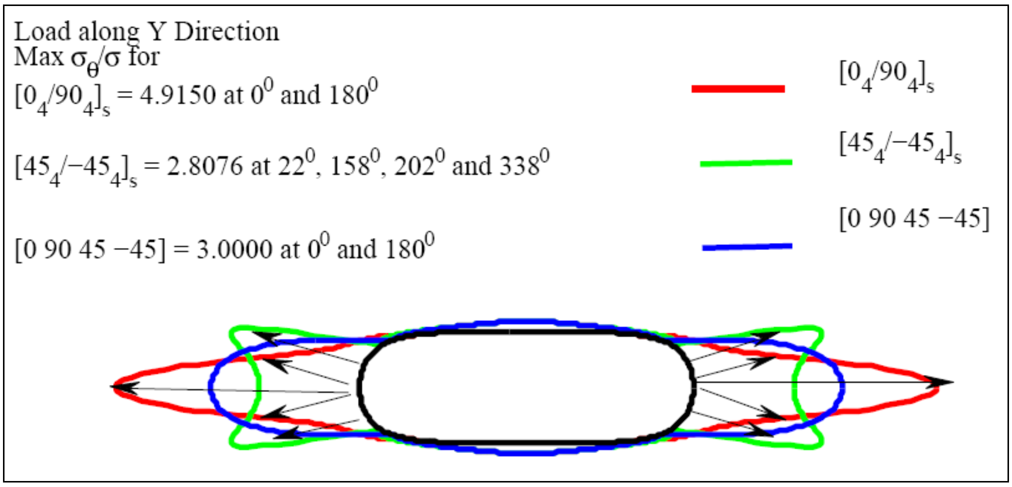

Figure 2. Effect of stacking sequence for load along Y axis

Figure 2 shows the normalized tangential stress for different stacking sequence for Graphite/Epoxy. The maximum values are 4.9150 and 3.0 at 0° and 180° for cross ply and unsymmetrical arrangement respectively. It is 2.8075 at 22°, 158°, 202° and 338° for angle ply arrangement. The values are high for cross ply arrangement and inclined fiber have less value of stress concentration.

4.2. Effect of Material Properties

The studies of the effect of materials are done with Graphite/Epoxy, Glass/Epoxy and Boron/Epoxy. The results are shown in Figure 3. The variation of normalized tangential stress is similar for all material, but Boron/Epoxy having highest value of 5.0977 at 90° and 270°.

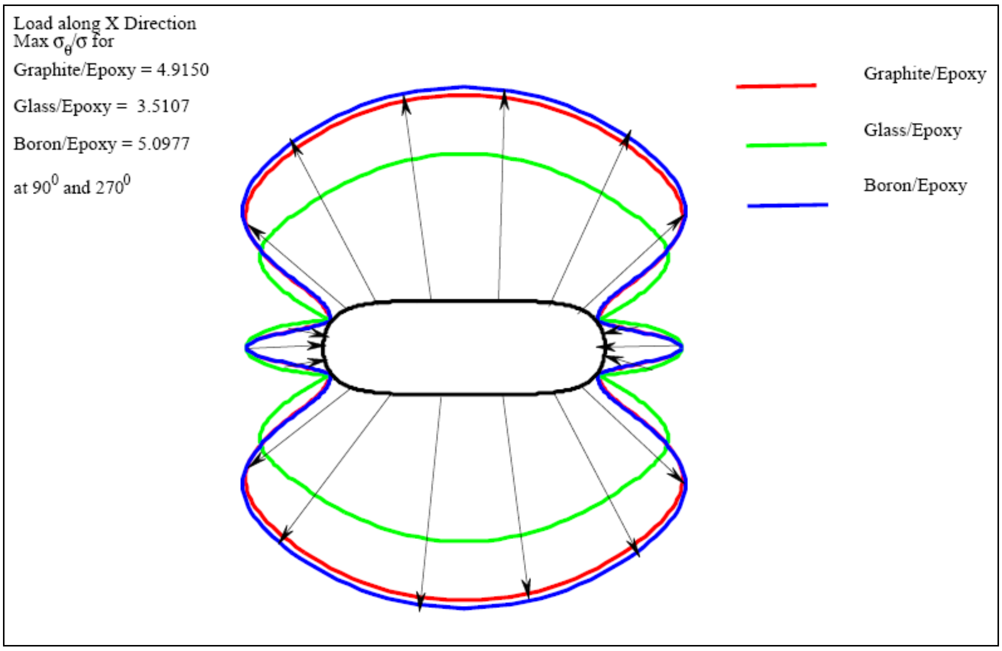

Figure 3. Effect of material properties

4.3. Effect of Load Angle

The effect of variation of load angle is obtained for Graphite/Epoxy material. The loading direction is initially along small axis of hole and varied through 90° to make it along long axis of the hole. The values of normalized tangential stress is obtained for cross ply $(0_4/90_4)_s$, angle ply $(45_4/\text{-}45_4)_s$ and unsymmetrical (0/90/45/-45) arrangement. Figure 4 shows the effect of load angle. The highest value is at 0° and 90° for cross ply and 54° for angle ply and unsymmetrical laminates. Hence, cross ply having highest value of stress concentration for uni-axial loading and angle ply having highest value of stress concentration for shear loading.

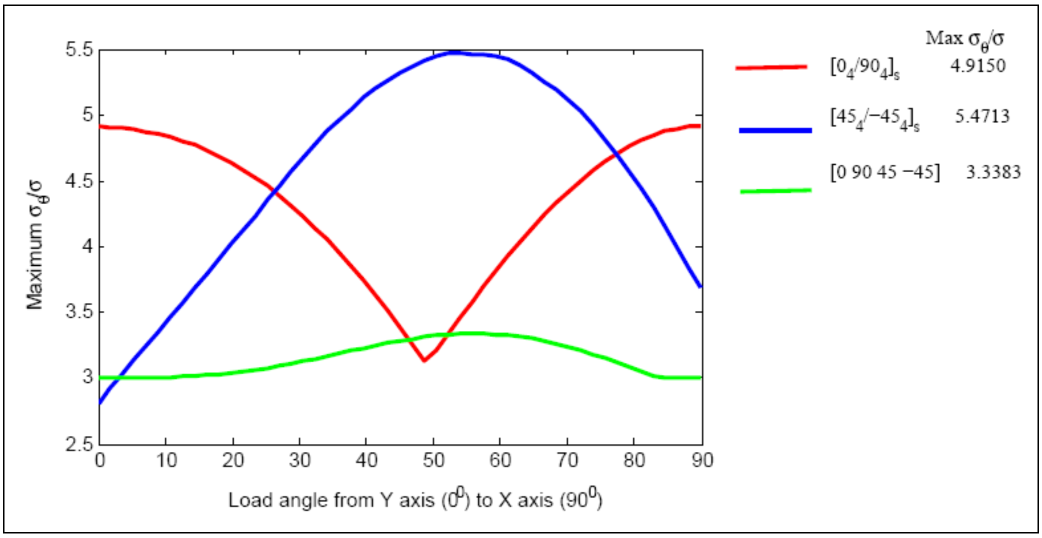

Figure 4. Effect of loading direction from Y axis to X axis

5. Conclusion

The presented solution can be used for isotropic and anisotropic material and it is handy tool to study the effect of various parameters on stress distribution. The mapping function is generalized to obtain different size of oval by varying value of constants c and ε. It observed that high value of stress concentration is obtained for cross ply arrangement for uni axial loading and angle ply for shear loading. It is also observed that stacking sequence and material properties have significant effect on stress distribution.

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