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Dynamic Finite Element Analysis of Laminated Composite Plates

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Abstract

Fibre Reinforced Laminated composites are widely used in Aerospace and Civil Engineering Structures due to its higher strength to weight and stiffness to weight ratio. This paper is concerned with the study of analysis and behaviour of laminated plated made up of continuous fibre reinforced composite materials. In analysis study natural frequencies and mode shapes are calculated using finite element method. Two Displacement models with six and eleven degrees of freedom per node based on Higher Order Shear Deformation Theory (HOSDT) are considered. First displacement model considers transverse displacement, rotations and their higher order terms. While second displacement model is based on inplane and transverse displacements, rotations and their higher order terms. Stiffness matrix and mass matrix of eight - node Quadrilateral isoparametric finite element is derived considering two displacement models. The objective is to study the performance of above derived finite elements in dynamic analysis of laminated composite plate. Free vibration analysis is carried out using finite element method to obtain natural frequencies and corresponding mode shapes with varying width-to-thickness ratio, material anisotropy, and number of layers of the fibers with different angle of orientation, support conditions and cut out. Further results of natural frequencies for several examples are compared with the results available in literatures for evaluating accuracy of displacement models. Computer programs are developed for automatic meshing of laminated composite plate, and for dynamic analysis to obtain natural frequencies and mode shapes.

Keywords: Laminated Composite Plate, Finite Element Method, Higher Order Shear Deformation Thoery, Dynamic Analysis

1. Introduction

In recent years, use of fibre reinforced laminated composite plates have been extensively increased as a standard segment, because of their appropriate properties such as high strength-to-weight ratio, high stiffness-to-weight ratio, etc. There are number of theories which have been developed including the effects of transverse shear deformations and transverse normal stresses and strains. The classical Kirchhoff thin plate theory (CLT), which ignores transverse shear effects. An improvement on the CLT is the first-order shear deformation theory (FSDT) such as the Reissner–Mindlin theory. Second and higher-order shear deformation plate theories use higher-order polynomials in the expansion of displacement components through the plate thickness and do not require shear correction factors.

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Nomenclature

a	length of plate
b	width of plate
h	thickness of plate
ω	non-dimensionalized natural frequency

2. A Higher Order Theory for Laminated Composites

Based on higher order theory formulated finite elements are to be applied to fibre reinforced composite laminated plates having thickness h in z-direction. Where the z = 0 is a reference plane located at the undeformed middle plane.

2.1. Displacement models

Displacement model is an assumption of a displacement components along x, y, z directions of the symmetrically laminated composite plates, which is to be applied in the finite element formulation. The considered diaplacement models are as follows,

Model-1:

$$U(x,y,z) = z\theta_x(x,y,0) + z^3\theta_x^*(x,y,0)$$
$$V(x,y,z) = z\theta_y(x,y,0) + z^3\theta_y^*(x,y,0)$$
$$W(x,y,z) = w_0(x,y,0) + z^2 w_0^*(x,y,0)$$

Model-2:

$$U(x,y,z) = u_0(x,y,0) + z\theta_x(x,y,0) + z^2u_0^*(x,y,0) + z^3\theta_x^*(x,y,0)$$
$$V(x,y,z) = v_0(x,y,0) + z\theta_y(x,y,0) + z^2v_0^*(x,y,0) + z^3\theta_y^*(x,y,0)$$
$$W(x,y,z) = w_0(x,y,0) + z\theta_z(x,y,0) + z^2 w_0^*(x,y,0)$$

Where $U(x,y,z)$, $V(x,y,z)$ and $W(x,y,z)$ express the displacements of a point (*x, y, z*) in the laminated plate along the *x*-, *y- and z-* directions respectively. Where u_0, v_0, w_0 are the associated mid-plane displacements. θ_x and θ_y are the rotations of the normal to the mid plane at the same point. $u_0^*, v_0^*, w_0^*, \theta_x^*$ and θ_y^* are the corresponding higher-order terms, and are also defined at the mid-plane.

2.2. Strain-Displacement Relations

The strain-displacement relations corresponding to the element nodal variables can be obtained as:

Displacement Model-1:

$$\varepsilon_x = z\,K_x + z^3K_x^*,\ \varepsilon_y = z\,K_y + z^3K_y^*,\ \varepsilon_z = z\,K_z^*,\ \gamma_{xy} = z\,K_{xy} + z^3K_{xy}^*,\ \gamma_{yz} = \emptyset_y + z^2\emptyset_y^*,\ \gamma_{xz} = \emptyset_x + z^2\emptyset_x^*$$

Displacement Model-2:

$$\varepsilon_x = \varepsilon_{0x} + z\,K_x + z^2\varepsilon_{0x}^* + z^3K_x^*,\quad \varepsilon_y = \varepsilon_{0y} + z\,K_y + z^2\varepsilon_{0y}^* + z^3K_y^*,\quad \varepsilon_z = \varepsilon_{0z} + z\,K_z^*$$
$$\gamma_{xy} = \varepsilon_{0xy} + z\,K_{xy} + z^2\varepsilon_{0xy}^* + z^3K_{xy}^*,\ \gamma_{yz} = \emptyset_y + z\psi_y + z^2\emptyset_y^*,\ \gamma_{xz} = \emptyset_x + z\psi_x + z^2\emptyset_x^*$$

Note that curvature quantities [K] and $[\emptyset]$ are associated with mid-plane quantities, hence at a cross-section they are constants throughout the depth.

2.3. Stress-strain relationship for laminated composite plates:

Unlike isotropic materials, anisotropic materials do not exhibit identical properties, thus under applied force stress-strain relationship can be obtained in matrix form as follows,

$[\sigma] = [C]\cdot[\varepsilon]$, where $[\sigma]$ is stress, $[\varepsilon]$ is strain matrix and $[C]$ is constitutive relation matrix.

The final constitutive matrix [C] is obtained by transforming the stresses and strains in principle material axis as follows,

$$[\sigma] = [T]^{-1}[C]\,[R]\,[T][R]^{-1}[\varepsilon]$$

Which can be written in compact form as follows,

$[\sigma] = [Q]\cdot[\varepsilon]$, where σ and ε stand for stress and strain vectors, respectively with reference to plate axes (x,y,z) (see

Fig.1).

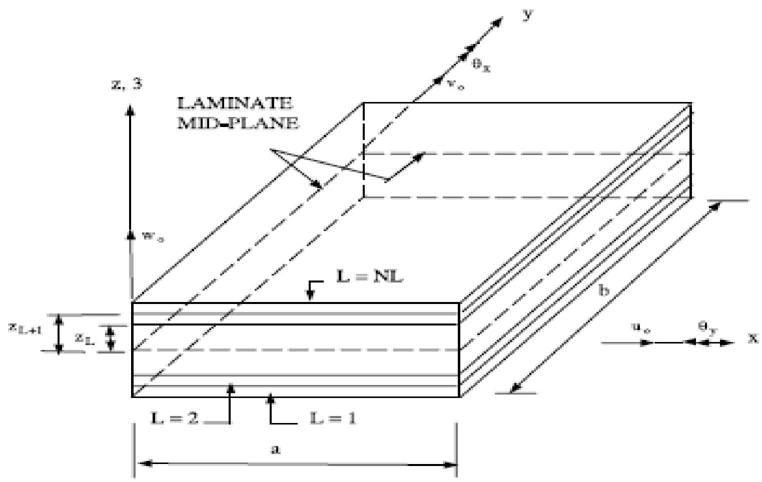


Fig. 1: Geometry of a four layer symmetric laminate

3. Finite Element Formulation:

The solution of the fundamental equations of the displacement model based on Higher Order Shear Deformation Theory (HOSDT) for anisotropic laminated plates, can conveniently be obtained by using the finite element displacement formulation. In present work, 8-node isoparametric quadrilateral element is used**.** The finite element formulation starts with writing the shape functions, followed by the derivation of the strain-displacement matrix [B], and calculation of element stiffness matrix. In formulation assumed displacement fields can be presented in relation with the corresponding nodal displacements using displacement shape functions, as shown below for displacement w_0 in z- direction,

$$w = \sum_{i=1}^{NN} N_i(x,y) \cdot w_{0i}$$

Where $N_i(x,y)$ is the shape function associated with node i, w_{0i} is the *z*- direction displacement of node i, and NN is the total number of nodes in the finite element. Similarly, the relation can be written fields θ_x, θ_y, w_0^*, θ_x^* and θ_y^***.**

3.1. Formulation of the element stiffness matrix

In derivation of stiffness matrix Virtual Work Principle has been used, which is denoted by $[K^e]$ the element stiffness matrix,

$$[K^e] = \{[B_b(x,y)]^T \{D_b\}[B_b(x,y)] + [B_s(x,y)]^T \{D_s\}[B_s(x,y)]\}\, dA$$

However, the shape functions and, thus, the matrices $[B_b]$ and $[B_s]$ are defined in terms of the non-dimensional coordinate system, the element stiffness matrix must be evaluated as follows,

$$[K^e] = \int_{-1}^{+1}\int_{-1}^{+1} \{[B_b]^T \{D_b\}[B_b] + [B_s]^T \{D_s\}[B_s]\}\ |J|\ d\xi\, d\eta\,,$$

Where J is the Jacobian matrix and $|J|$ defines the determinant of the Jacobian matrix. The Gauss-Quadrature integration technique has been used to evaluate the integrals. In the present formulation bending and shear stiffness matrix have been evaluated using 3 x 3 integration scheme. Thus the stiffness matrix can be written as follows,

$$[K^e] = \sum_{a=1}^{NGB}\sum_{b=1}^{NGB} \{[B_b]^T\{D_b\}[B_b]\}\,|J|\,W_a W_b W_c \;+\; \sum_{a=1}^{NGS}\sum_{b=1}^{NGS} \{[B_s]^T\{D_s\}[B_s]\}\,|J|\,W_a W_b W_c\,,$$

Where W_a, W_b and W_c are the weighting factors corresponding to the Gauss sampling points and NGB and NGS define the number of Gauss points selected for the integration schemes for bending stiffness and shear stiffness, respectively.

3.2. Formulation of the element mass matrix

In derivation of Mass matrix the Hamilton's Principle has been used. Here $[M^e]$ element mass matrix is given by,

$$M^e = [\int \rho\, N^T m\, N\ dV]$$

Where m and N is inertia matrix and shape function respectively.

In which $N = [N_i : \dots\dots\dots : N_{NN}]$ and m is given by $[I_1, I_2, I_3, I_4, I_5, I_6] = \sum_{L=1}^{n} \int_{h_{l-1}}^{h_l} [1, z^2, z^4, z^6]\ \rho^t dz$, where ρ^t is material density and I_1, I_2, I_4, I_6 is the normal inertia, rotary inertia and higher order inertia terms respectively. However the shape functions are in non-dimensional coordinate system, the element mass matrix must be evaluated as follows,

$$[M^e] = \int_{-1}^{+1}\int_{-1}^{+1} [N]^T [m][N]\ J\ d\xi\ d\eta$$

Where J is the Jacobian matrix and $|J|$ defines the determinant of the Jacobian matrix. The Gauss-Quadrature integration technique is used to evaluate the integrals. In the present formulation element mass matrix is evaluated using 3x3 integration scheme. Thus the mass matrix can be written as follows,

$$[M^e] = \sum_{a=1}^{NG}\sum_{b=1}^{NG} [N]^T [m][N]\ |J|\ W_a W_b$$

Where W_a, W_b are the weighting factors corresponding to Gauss sampling points and NG define the number of Gauss points selected for the integration schemes.

4. Discretization of Problem:

A computer program incorporating present higher-order theory is developed for free vibration analysis of a laminated composite plate without cut-out (see **Fig. 2**) and with cut-out at centre (see **Fig. 3**). In general all problems are solved by using 4x4 mesh size in quarter part of plate and natural frequencies and mode shapes are found out by using developed program for specific boundary conditions as shown in **Fig. 4**.

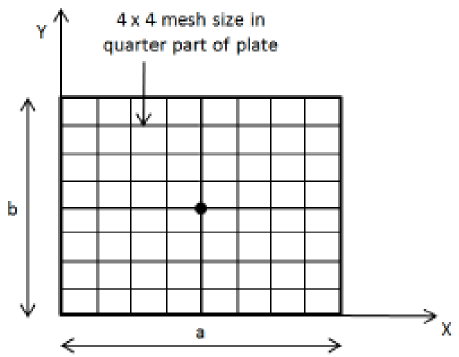


Fig. 2: Meshing in quarter part of plate

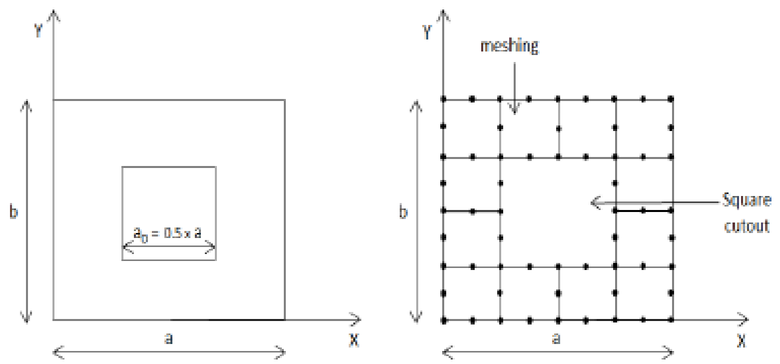
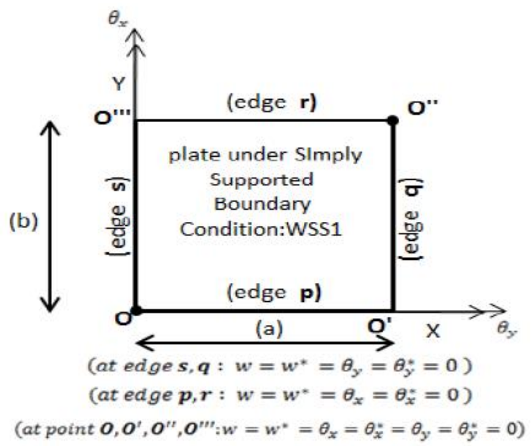


Fig. 3: plate with square cut-out at centre

Material Sets considered in analysis of problems:
Material set 1: $E_1/E_2 = 40,\ E_2 = E_3,\ G_{12} = G_{13} = 0.6E_2,\ G_{23} = 0.5E_2,\ \vartheta_{12} = \vartheta_{23} = \vartheta_{13} = 0.25$
Material set 2: $E = 200000,\ \vartheta = 0.3,\ \rho = 0.08,\ G = \frac{E}{2(1+\vartheta)},\ D = \frac{E\,h^3}{12(1-\vartheta^2)}$



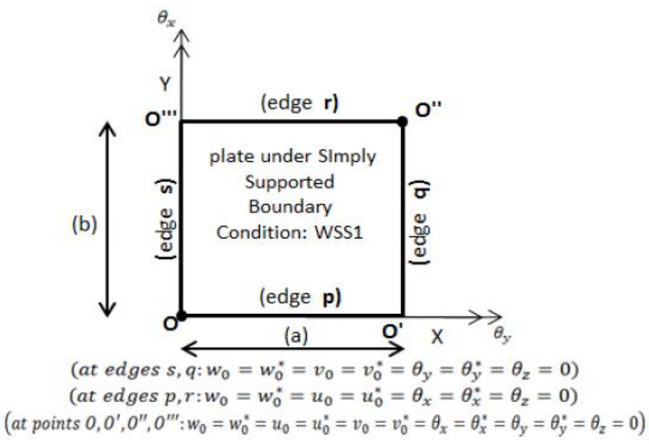


Fig.4. (a) Plate under simply supported boundary condition model-1 **(b)** Plate under simply supported boundary condition for model-2

5. Results and Discussion

Example -1: Non-dimensionalized frequency is calculated for a plate having a symmetric cross-ply stack-up as shown in **Fig.5**, with equal layer thickness, Material set-1, with simply supported along all four edges. Table 1 shows comparison of non-dimensionalized fundamental frequency obtained by various theories for laminated plate without cut-out.

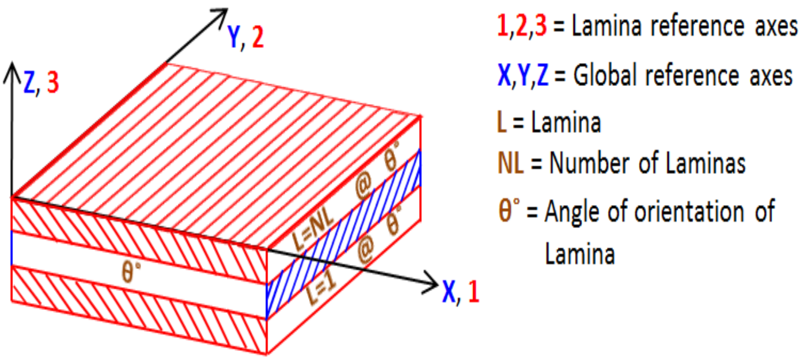


Fig.5. Cross – ply laminated composite plate

Table 1: Comparison of non-dimensionalized fundamental frequency $\bar{\omega} = \omega\sqrt{\rho h^2/E_2}$ for a symmetric simply supported cross-ply square laminated plates with a/h= 5.

Layers	Source	a/h				
		5	**10**	**20**	**50**	**100**
$(0^o/90^o)_1$	Present Model 1	0.471052	0.159505	0.044992	0.007495	0.001885
	Present Model 2	0.348600	0.104418	0.027830	0.004593	0.001171
	PHOST 11	0.348106	0.104157	0.027651	0.004507	0.001129
	PHOST 6	0.469961	0.159299	0.044969	0.007495	0.001885
$(0^o/90^o)_2$	Present Model 1	0.471052	0.159505	0.044992	0.007495	0.001885
	Present Model 2	0.432900	0.146500	0.044240	0.007545	0.001925
	PHOST 11	0.432405	0.146309	0.041238	0.006868	0.001727
	PHOST 6	0.469961	0.159299	0.044969	0.007495	0.001885
$(0^o/90^o)_3$	Present Model 1	0.471052	0.159505	0.044992	0.007495	0.001885
	Present Model 2	0.45200	0.153600	0.043520	0.007312	0.001860
	PHOST 11	0.451414	0.153371	0.043329	0.007222	0.001817
	PHOST 6	0.469961	0.159299	0.044969	0.007495	0.001885
$(0^o/90^o)_5$	Present Model 1	0.471052	0.159505	0.044992	0.007495	0.001885
	Present Model 2	0.463500	0.157400	0.044520	0.007486	0.001906
	PHOST 11	0.462951	0.157158	0.044382	0.007397	0.001861
	PHOST 6	0.469961	0.159299	0.044969	0.007495	0.001885

Example -2: Non-dimensionalized frequency is calculated for a isotropic plate having a square cut out at centre as shown in **Fig.3** for h/a = 0.01 considering material set-2. Table 2 shows comparison of non-dimensionalized fundamental frequency obtained for laminated plate with cut-out at centre.

Table 2: Comparison of non-dimensionalized fundamental frequency$\bar{\omega} = \sqrt[4]{\rho h\, \omega^2 a^4 / D(1-\vartheta^2)}$ for a isotropic simply supported plate with square cut out at centre for h/a = 0.01.

Mode	$\bar{\omega}$		
	Present Model 2		RPIM with cut out
	Without cut out	With cut out	
1	4.6640	5.0120	4.9217
2	7.8671	7.1355	6.4810

3	7.9654	7.1355	6.4821
4	11.8302	9.4337	8.5509

Conclusion

A Higher Order Shear Deformation Theory is used for static analysis of laminated composite plates. To derive 8-node isoparametric finite element formulation two displacement models with six and eleven degrees of freedom per node based on higher order theory is used. To validate the finite element formulation comparison study of obtained numerical results is carried out which shows the following results,

(1). In the case of solution with displacement model-1, which includes bending action only, as the a/h ratio increase the effect of the coupling between bending and stretching increases for two layers and four layers. The percentage errors in natural frequency of laminated composite plate are as high as 67% for cross-ply and 75.8% for angle-ply. The percentage error decreases with the increase in number of layers. Therefore it is concluded that the coupling between bending and stretching has a significant effect on the behaviour of antisymmetric laminates with few lamina. So displacement model - 1 can be considered for vibration analysis of laminated composite plate having symmetrical lamina orientation but cannot be used for unsymmetrical orientation. For antisymmetric orientation of lamina displacement model - 2 should be used, for obtaining natural frequencies and mode shapes.

(2). It is observed that for laminated plate with square cut out of $a_0 = 0.5*a$ size at centre, results of natural frequencies are overestimated by 8.01% while comparing those with literature. Also the rate of increase of natural frequency is more in case of plate without cut out compare to plate with square cut out at centre.

From this study it can also be concluded that a 4x4 uniform mesh in quarter plate can be used to obtain sufficient accurate result. From this study it can also be concluded that the developed computer program is capable to carryout static analysis of laminated composite plates with varying width-to-thickness ratio, material anisotropy, number of layers and support conditions.

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